CERTAIN CYCLIC INVOLUTORY MAPPINGS ON HYPERSPACE SURFACES

By

STANLEY FRANK

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA
January, 1960

ACKNOWLEDGMENTS

We live in a world where the love of truth and a sense of dedication to one's profession are still too rare to be taken for granted. Dr. W. R. Hutcherson, chairman of the author's supervisory committee, exemplifies these qualities to a high degree. His spiritual strength, combined with the sharing of his rich mathematical experience, has gone far in making this dissertation possible.

The author was exceedingly fortunate to have on his supervisory committee Dr. T. O. Moore, Dr. A. Sobczyk, Dr. D. E. South, and Dr. H. Lewis. Each of these men possesses the wonderful attribute of not knowing how to say "no" to a student in need of instruction or guidance.

TABLE OF CONTENTS

		Page
ACKNOWLEDGMENTS		ii
Chapter I.	INTRODUCTION	1
II.	CUBIC PLANE CURVES MAPPED ON A CUBIC SURFACE IN ORDINARY 3-SPACE, S_3	3
III.	SEVENTH DEGREE PLANE CURVES MAPPED ON A RATIONAL SURFACE IN S_6	6
IV.	A QUARTIC SURFACE IN S_3 MAPPED ON A SURFACE IN S_{38}	11
V.	MAPPING OF CERTAIN PLANE ELEVENTH DEGREE CURVES INVARIANT UNDER I_{11}	21
VI.	A STUDY OF BRANCH POINT O_1'	28
VII.	A STUDY OF BRANCH POINT O_8'	65
VIII.	A STUDY OF BRANCH POINT O_7'	95
APPENDICES		
I.	A METHOD FOR FINDING THE ORDER OF SURFACE F	120
II.	A NOTE ON POINTS OF COINCIDENCE AND PERFECT POINTS	122
BIBLIOGRAPHY		123
BIOGRAPHICAL SKETCH		125

CHAPTER I

INTRODUCTION

This dissertation, for the purposes of classification, may be divided into two sections. The first section comprises the first three chapters (after the introduction), in which three relatively independent studies are conducted. Chapters V, VI, VII, and VIII constitute the second, and essentially original, section.

One central theme which runs thru classical projective geometry is the mapping of algebraic systems of curves onto surfaces in hyperspace. Under such mappings, certain unusual situations arise which demand special treatment. One such situation, which arises frequently in the present paper, is the appearance of a branch point when a (1,1) correspondence is desired between two sets of points. Chapters VI, VII, and VIII deal specifically with the study of such branch points. Complete summaries of the results obtained in these chapters are located at the end of each chapter.

In Chapter II, a system of cubic plane curves is mapped on a cubic surface ϕ in S_3. This system, designated by $|C|$, is invariant under the homography

T, $\quad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^2x_3,$

where $E^3 = 1$ [6]. A second invariant system, $|C_1|$, is mapped onto the surface ϕ, and its image curves, A, on ϕ in S_3 are studied. Finally, by use of a quadratic transformation, the branch point O_1' is studied.

Chapter III treats of the mapping of seventh degree plane curves on a rational surface in S_6, under the homography

H, $\qquad x_1' : x_2' : x_3' : = x_1 : Ex_2 : E^2x_3,$

where $E^7 = 1$. First an invariant system of the seventh degree plane curves is mapped onto a surface ϕ in S_5. Then, by an interesting technique, the same system is mapped onto a surface F in S_6. Finally the surface F is shown to be rational.

The second and third chapters deal with the mapping of systems of curves onto surfaces. In Chapter IV, a quartic surface F in S_3, which is invariant under a cyclic collineation

T, $\qquad x_1' : x_2' : x_3' : x_4' = x_1 : Ex_2 : E^5x_3 : x_4,$

where $E^{11} = 1$, is mapped onto a surface in S_{38}.

The vertices of the tetrahedron of reference are invariant under T and lie on F. These four vertices are studied next, and the edges of the tetrahedron of reference are also studied. After the mapping is carried out, a plane quadratic transformation is used to study the characteristics of the vertex P_1. Finally, a number of theorems are developed concerning the existence of perfect† points in the neighborhoods of P_1.

Homogeneous coordinates [8] are used throughout this dissertation. If the reader desires an explanation of "first order neighborhood," he is referred to Morelock [13].

---

†See Appendix II.

CHAPTER II

CUBIC PLANE CURVES MAPPED ON A CUBIC SURFACE IN ORDINARY 3-SPACE, S_3

The homography T, $x_1' : x_2' : x_3' = x_1 : Ex_2 : E^2x_3$, where E is a primitive cube root of unity, generates an involution I_3 of period three. The three linear systems of plane curves

$$|C|, \qquad v_1x_1^3 + v_2x_1x_2x_3 + v_3x_2^3 + v_4x_3^3 = 0$$
$$|C_1|, \qquad u_1x_1^2x_3 + u_2x_1x_2^2 + u_3x_2x_3^2 = 0$$
$$|C_2|, \qquad u_1x_1^2x_2 + u_2x_2^2x_3 + u_3x_1x_3^2 = 0$$

are invariant under T [5,6]. A curve of $|C|$ will be designated by C and a similar notation will hold for the other systems. A curve C has no base points, but a curve C_1 has three base points $O_1(1,0,0)$, $O_2(0,1,0)$, and $O_3(0,0,1)$. We relate projectively the curves C to the planes of an ordinary space of three dimensions, S_3, by taking

$$(T_1) \qquad \frac{X_1}{x_1^3} = \frac{X_2}{x_1x_2x_3} = \frac{X_3}{x_2^3} = \frac{X_4}{x_3^3}.$$

Eliminating x_1, x_2, and x_3 in T_1, we obtain the cubic surface

$$\phi, \qquad X_2^3 = X_1X_3X_4.$$

This surface is the image of I_3 and is of order three. Also, we note that the curves C_1 pass simply thru O_1, O_2, and O_3.

Each curve C_1 is tangent to the line $x_3 = 0$ at O_1, to the line $x_1 = 0$ at O_2, and to the line $x_2 = 0$ at O_3.

It will be our aim here to study the behavior of the image curves in S_3 of the curves C_1. These image curves form a system of curves which we shall designate by $|A|$. Any one of the curves of $|A|$ will be designated by A. The curves A lie on ϕ.

We are especially interested in the behavior of the curves A in the domain of the points $O_1'(1,0,0,0)$, $O_3'(0,0,1,0)$, and $O_4'(0,0,0,1)$, which are the image points of O_1, O_2, and O_3 respectively.

The quadratic transformation

$$(T_2) \qquad x_1 : x_2 : x_3 = y_1^2 : y_1y_2 : y_2y_3$$

makes the point $y_2 = y_3 = 0$ correspond to the point O_{12} which lies on the line $x_3 = 0$ infinitely near O_1 [4,7]. If we apply this transformation to the equation of the curves C which pass thru O_1, we obtain

$$(v_2y_3 + v_3y_2)y_1^3 + v_4y_2y_3^3 = 0.$$

Hence, the curves C which pass thru O_1 pass simply thru O_{12}. From T_1 we see that

$$\frac{X_2}{y_1^3y_3} = \frac{X_3}{y_1^3y_2} = \frac{X_4}{y_2y_3^3}.$$

By placing $y_3 = ky_2$ and letting y_2 tend to 0, we obtain the parametric equations

$$\frac{X_2}{k} = \frac{X_3}{1}, \quad X_4 = 0.$$

This is one of the tangent planes to surface ϕ at O_1'.

Now, applying the quadratic transformation T_2 to the equation of curves C_1 we obtain

$$(u_1y_3 + u_2y_2)y_1^3 + u_3y_2^2y_3^2 = 0.$$

Again, placing $y_3 = ky_2$ and letting y_2 tend to zero we have

$$u_1k + u_2 = 0.$$

Solving this equation for k and substituting in the equation of the tangent plane we get

$$u_1X_2 + u_2X_3 = 0,\ X_4 = 0.$$

This is the tangent element to a curve A at O_1'. Since this tangent element is a line, we see that the curves A pass simply thru O_1'.

We could determine the tangent elements to the curves A at O_3' and O_4' in a similar fashion. Using the quadratic transformation

$$(T_3) \qquad x_1 : x_2 : x_3 = y_1y_3 : y_2^2 : y_2y_3$$

the equation of the tangent line to the curve A at O_3' is

$$u_2X_2 + u_3X_4 = 0$$

$$X_1 = 0.$$

Application of the quadratic transformation

$$(T_4) \qquad x_1 : x_2 : x_3 = y_1y_3 : y_1y_2 : y_3^2$$

leads to the equation of the tangent line to the curve A at O_4' which is

$$u_1X_1 + u_3X_2 = 0$$

$$X_3 = 0.$$

CHAPTER III

SEVENTH DEGREE PLANE CURVES MAPPED ON A RATIONAL SURFACE IN S_6

Let us consider the homography

$$(H) \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^2x_3 \qquad (E^7 = 1)$$

It engenders [8,10] an involution I_7 of period seven. This involution I_7 has for its base points $O_1(1,0,0)$, $O_2(0,1,0)$, and $O_3(0,0,1)$.

Seven seventh order curves which are invariant under this homography H follow:

$$[1] \qquad v_1x_1^7 + v_2x_2^7 + v_3x_3^7 + v_4x_1x_2^5x_3 + v_5x_1^5x_2x_3^5 + v_6x_1^2x_2^5x_3^2 = 0$$

$$[2] \qquad u_1x_1^6x_2 + u_2x_2^6x_3 + u_3x_1x_2^4x_3^2 + u_4x_1^3x_3^4 + u_5x_1^2x_2^2x_3^3 = 0$$

$$[3] \qquad a_1x_1^6x_3 + a_2x_1^5x_2^2 + a_3x_2^5x_3^2 + a_4x_1^2x_2^1x_3^4 + a_5x_1x_2^3x_3^3 = 0$$

$$[4] \qquad b_1x_1^5x_2x_3 + b_2x_1^2x_3^5 + b_3x_1^4x_2^3 + b_4x_2^4x_3^3 + b_5x_1x_2^2x_3^4 = 0$$

$$[5] \qquad c_1x_1^5x_3^2 + c_2x_1x_2x_3^5 + c_3x_1^4x_2^2x_3 + c_4x_1^3x_2^4 + c_5x_2^3x_3^4 = 0$$

$$[6] \qquad d_1x_1x_3^6 + d_2x_1^2x_2^5 + d_3x_2^2x_3^5 + d_4x_1^4x_2x_3^2 + d_5x_1^3x_2^3x_3 = 0$$

$$[7] \qquad e_1x_1x_2^6 + e_2x_2x_3^6 + e_3x_1^4x_3^5 + e_4x_1^2x_2^4x_3 + e_5x_1^3x_2^2x_3^2 = 0.$$

Only [1] lacks base points. The others have O_1, O_2, and O_3 as base points.

Let us now transform projectively the curves [1] to hyperplanes of a linear space of five dimensions, S_5, by use of

$$(T) \qquad \frac{X_1}{x_1^7} = \frac{X_2}{x_2^7} = \frac{X_3}{x_3^7} = \frac{X_4}{x_1x_2^5x_3} = \frac{X_5}{x_1^5x_2x_3^5} = \frac{X_6}{x_1^2x_2^5x_3^2}$$

Upon eliminating x_1, x_2, and x_3 in T we obtain a surface

$$\phi, \qquad \begin{Vmatrix} X_5^2 & X_1X_6 & X_1X_4 & X_4^3X_5 \\ X_3 & X_5 & X_6 & X_2^2X_3 \end{Vmatrix} = 0.$$

Consider the surface

$$F, \qquad \begin{Vmatrix} X_5^2 & X_1X_6 & X_1X_4 & X_4^3X_5 \\ X_3 & X_5 & X_6 & X_2^2X_3 \end{Vmatrix} = 0.$$

$$X_7^7 = \Psi(X_1, X_2, X_3, X_4, X_5, X_6)$$

where Ψ is defined as the seventh power of the left member of equation [2].

We shall now show that F is a rational surface. Take

$$R, \qquad \frac{X_1}{x_1^7} = \frac{X_2}{x_2^7} = \frac{X_3}{x_3^7} = \frac{X_4}{x_1x_2^5x_3} = \frac{X_5}{x_1^3x_2x_3^3} = \frac{X_6}{x_1^2x_2^3x_3^2}$$

$$= \frac{X_7}{u_1x_1^6x_2 + u_2x_2^6x_3 + u_3x_1x_2^4x_3^2 + u_4x_1^3x_3^4 + u_5x_1^2x_2^2x_3^3} = \frac{1}{\lambda}.$$

This projection causes a point of the plane to correspond to a point of F. We are going to show that the inverse is equally true. Let us

deduce the first of the following equations from equation R. We may then obtain the others by multiplying the two members of the preceding equation by $X_4X_5X_6$ and applying T.

[8] $u_1x_1^6x_2 + u_2x_2^6x_3 + u_3x_1x_2^4x_3^2 + u_4x_1^5x_3^4 + u_5x_1^2x_2^2x_3^3 = \lambda X_7.$

[9] $u_1x_1^3x_3^4X_1X_2X_6 + u_2x_1^6x_2X_2^2X_3 + u_3x_2^6x_3X_1X_2X_5$

$+ u_4x_1^2x_2^2x_3^5X_1X_2X_3 + u_5x_1x_2^4x_3^2X_1X_2X_3 = \lambda X_7(X_4X_5X_6).$

[10] $u_1x_1^5x_3^4x_1^2x_2^2X_3X_4 + u_2x_1^2x_2^3x_1^5X_3X_5 + u_3x_1^6x_2X_1x_2^5x_3^2$

$+ u_4x_1x_2^4x_3^2x_1^2x_2^2x_3^2 + u_5x_2^6x_3x_1^2x_2^2x_3^2 = \lambda X_7(X_4^2X_5^2X_6^2).$

[11] $u_1x_1^5x_3^4x_1^3x_2^4x_3^2 + u_2x_1x_2^4x_3^2x_1^2x_2^4x_3^2X_5 + u_3x_1^2x_2^2x_3^5x_1^2x_2^4x_3^2X_5$

$+ u_4x_2^6x_3x_1^3x_2^3x_3^3 + u_5x_1^6x_2x_1^2x_2^4x_3^5 = \lambda X_7(X_4^3X_5^3X_6^3).$

[12] $u_1x_1^3x_3^4x_1^3x_2^5x_3^2x_5^2 + u_2x_2^6x_3x_1^5x_2^3x_3^3X_5 + u_3x_1x_2^4x_3^2x_1^5x_2^3x_3^3X_5$

$+ u_4x_1^6x_2x_1^3x_2^5x_3^4 + u_5x_1^2x_2^2x_3^3x_1^5x_2^3x_3^3X_5 = \lambda X_7(X_4^4X_5^4X_6^4).$

[13] $u_1x_1^5x_3^4x_1^4x_2^6x_3^5X_5X_6 + u_2x_1^6x_2x_1^3x_2^7x_3^4X_5 + u_3x_2^6x_3^5X_1^4x_2^6x_3^4X_5$

$+ u_4x_1^2x_2^2x_3^5x_1^4x_2^6x_3^4X_5 + u_5x_1x_2^4x_3^2x_1^4x_2^6x_3^4X_5 = \lambda X_7(X_4^5X_5^5X_6^5).$

When we examine these six equations, we find that in the coefficients of the u_i (i = 1,2,3,4,5), appear certain combinations of x_1, x_2, x_3 in each equation which follows. If we consider these combinations as unknowns, we can solve the above set of equations in terms of u_i and X_1, obtaining

$$x_1^6 x_2 = \lambda X_7 \frac{\Delta_1}{\Delta} \qquad x_1^5 x_3^4 = \lambda X_7 \frac{\Delta_4}{\Delta}$$

$$x_2^6 x_3 = \lambda X_7 \frac{\Delta_2}{\Delta} \qquad x_1^2 x_2^2 x_3^3 = \lambda X_7 \frac{\Delta_5}{\Delta} .$$

$$x_1 x_2^4 x_3^2 = \lambda X_7 \frac{\Delta_3}{\Delta}$$

We also deduce from R

$$x_1^7 = \lambda X_1 \qquad x_1 x_2^5 x_3 = \lambda X_4$$

$$x_2^7 = \lambda X_2 \qquad x_1^3 x_2 x_3^3 = \lambda X_5$$

$$x_3^7 = \lambda X_3 \qquad x_1^2 x_2^3 x_3^2 = \lambda X_6 .$$

Upon combining these equations with the preceding solutions we obtain

$$R^{-1}, \quad \frac{x_1}{X_4 X_5 \Delta^2} = \frac{x_2}{X_4 X_7 \Delta_5 \Delta} = \frac{x_3}{X_7^2 \Delta_3 \Delta_5} .$$

Hence we have (1,1) correspondence between points on the plane and points on surface F.

By use of the homography

$$x_1' : x_2' : x_3' = x_1 : Ex_2 : E^2x_3$$

we notice that the points on the plane

$$x_1 : E^2x_2 : E^4x_3$$

$$x_1 : E^3x_2 : E^6x_3$$

$$x_1 : E^4x_2 : Ex_3$$

$$x_1 : E^5x_2 : E^3x_3$$

$$x_1 : E^6x_2 : E^5x_3$$

correspond to the following points on F, respectively, and conversely:

$$X_1 : X_2 : X_3 : X_4 : X_5 : X_6 : E^2X_7$$

$$X_1 : X_2 : X_3 : X_4 : X_5 : X_6 : E^3X_7$$

$$X_1 : X_2 : X_3 : X_4 : X_5 : X_6 : E^4X_7$$

$$X_1 : X_2 : X_3 : X_4 : X_5 : X_6 : E^5X_7$$

$$X_1 : X_2 : X_3 : X_4 : X_5 : X_6 : E^6X_7 .$$

CHAPTER IV

A QUARTIC SURFACE IN S_3 MAPPED ON A SURFACE IN S_{58}

In this chapter, a quartic surface will be considered. It contains a cyclic involution of period eleven [10, 11].

The surface

$$F_4(x_1,x_2,x_3,x_4) = ax_1x_2x_3^2 + bx_1^2x_4^2 + cx_1^3x_4 + dx_4^4 = 0$$

is invariant under a cyclic collineation T of period eleven,

$$x_1' : x_2' : x_3' : x_4' = x_1 : Ex_2 : E^5x_3 : x_4 \qquad (E^{11} = 1).$$

The points $P_1(1,0,0,0)$, $P_2(0,1,0,0)$, and $P_3(0,0,1,0)$ are all invariant under T and lie on the surface F_4. The point $P_4(0,0,0,1)$ is invariant; however, it does not lie on F_4. These facts may be stated in the following theorem.

THEOREM 1. Each vertex of the tetrahedron of reference is a point of coincidence and, with the exception of P_4, lies on the surface.

The surface F_4 passes thru the edge P_2P_3(for $x_1 = x_4 = 0$). It also passes thru the edges $P_1P_3(x_2 = x_4 = 0)$ and $P_1P_2(x_3 = x_4 = 0)$.

It is easily verified that F_4 does not pass thru the remaining three edges. A second theorem may now be stated.

THEOREM 2. The surface F_4 includes three edges of the tetrahedron of reference; namely, P_2P_3, P_1P_3, and P_1P_2. It does not contain the remaining three edges, P_1P_4, P_2P_4, and P_3P_4.

It is to be noted that P_1 is simple on F_4, P_2 is triple, P_3 is double, while P_4 does not lie on the surface. Only P_1 will be investigated in detail at this time.

Consider a curve C, not transformed into itself by T, passing thru P_1 and lying on F_4. Take the plane $x_2 + kx_3 = 0$ of the pencil passing thru P_1 and P_4, tangent to C. This plane is transformed into $Ex_2 + kE^5x_3 = 0$ or $x_2 + kE^4x_3 = 0$ by T and hence is non-invariant. The curve cut out on F_4 by $x_2 + kx_3 = 0$ is therefore non-invariant. The common tangent to the two curves is not transformed into itself. Thus C and its image do not touch each other at P_1. Now, since C was a variable curve thru P_1, satisfying the non-invariant property, it follows that P_1 is an imperfect† coincidence point.

Let us now consider the points P_2 and P_3. We wish to find whether or not they are perfect. Consider a curve C_1, not transformed into itself by T, passing thru P_2 and lying on F_4. Note that $x_1 = 0$ is one tangent plane and $x_3 = 0$ is a tangent plane taken twice.

---

†See Appendix II.

For the first branch, take the plane $x_3 + kx_4 = 0$ of the pencil passing thru P_1 and P_2, tangent to C_1. The plane is transformed into $E^5x_3 + kx_4 = 0$ by T and hence is non-invariant. The curve cut out on F_4 by $x_3 + kx_4 = 0$ is therefore non-invariant. The common tangent to the two curves is not transformed into itself. Thus C_1 and its image along the first branch do not touch each other at P_1. By an argument similar to that for C and the study of P_1, it follows that P_2 is an imperfect point of coincidence along the first branch.

For the second branch, take the plane $x_1 + kx_4 = 0$ of the pencil passing thru P_2 and P_3, tangent to C_1. Since this plane is invariant under T, the curve cut out on F_4 by $x_1 + kx_4 = 0$ is invariant. Thus the curve C_1 and its image along the second branch touch each other at P_2. Hence, there is a perfect point along this branch.

Again, consider a curve C_2, not transformed into itself by T, passing thru P_3 and lying on F_4. Note that $x_1 = 0$ and $x_2 = 0$ are tangent planes at P_3. For the first branch, take the plane $x_2 + kx_4 = 0$ of the pencil passing thru P_1 and P_3, tangent to C_2. Since this plane is transformed into $Ex_2 + kx_4 = 0$ and is thus non-invariant, one uses the same argument as in the case of P_2 to show that P_3 is imperfect on the plane $x_1 = 0$. For the second branch, the plane $x_1 + kx_4 = 0$ is transformed under T into itself. Hence, P_3 is a perfect point along this branch.

THEOREM 3. In the I_{11} belonging to F_4 in S_3, P_1 is an imperfect coincidence point, while P_2 and P_3 are both imperfect along one

of their branches and perfect points of coincidence along their other branches.

Consider the complete system of curves $|A|$ cut out on F_4 by all surfaces of order eleven. Its dimension is 243, its genus is 245, and the number of variable intersections of two members of the system is 484. A curve A of this system is not in general transformed into itself by T. There are, however, eleven partial systems $|A_1|$ in $|A|$ which are transformed into themselves. By use of $|A_1|$ we find

$$a_1x_1^{11} + a_2x_2^{11} + a_3x_3^{11} + a_4x_4^{11} a_5x_1^{10}x_4 + a_6x_1x_4^{10}$$

$$+ a_7x_1^9x_4^2 + a_8x_1^2x_4^9 + a_9x_1^8x_4^3 + a_{10}x_1^8x_2^2x_3 + a_{11}x_1^3x_4^8$$

$$+ a_{12}x_2^2x_3^8x_4 + a_{13}x_1^7x_2^2x_3^2x_4 + a_{14}x_1^7x_4^4 + a_{15}x_1^7x_2^3x_3 + a_{16}x_2^7x_3^3x_4$$

$$+ a_{17}x_1^4x_4^7 + a_{18}x_1x_2^2x_3^2x_4^7 + a_{19}x_1^6x_2^2x_3^2x_4^2 + a_{20}x_1^6x_4^5 + a_{21}x_1^4x_2^6x_3$$

$$+ a_{22}x_1^3x_2^6x_3^6x_4 + a_{23}x_2^6x_3^5x_4^4 + a_{24}x_1^6x_2^6x_3^5x_4^3 + a_{25}x_1^2x_2^6x_3^5x_4^2 + a_{26}x_1^2x_2^3x_3^6$$

$$+ a_{27}x_2^3x_3^6x_4^2 + a_{28}x_1^3x_2^3x_3^6x_4 + a_{29}x_1^5x_4^6 + a_{30}x_1^2x_2^2x_3^2x_4^6 + a_{31}x_1^5x_2^2x_3^4$$

$$+ a_{32}x_1^5x_2^2x_3^2x_4^3 + a_{33}x_2^2x_3^4x_4^5 + a_{34}x_1^4x_2^2x_3^4x_4 + a_{35}x_1^4x_2^2x_3^4$$

$$+ a_{36}x_1^2x_2^3x_3^4x_4^4 + a_{37}x_1^2x_2^2x_3^4x_4^3 + a_{38}x_1^3x_2^2x_3^4x_4^2 + a_{39}x_1^3x_2^2x_3^5x_4^5 = 0.$$

We refer the curves A_1 projectively to the hyperplanes of a linear space of 38 dimensions. We obtain a surface Φ of order 44,

as the image of I_{11}. The equations for the mapping of I_{11} upon ϕ in S_{38} are given by the following transformation:

$$\lambda X_1 = x_1^{11} \qquad \lambda X_{14} = x_1^7 x_4^4 \qquad \lambda X_{27} = x_2^3 x_3^6 x_4^2$$

$$\lambda X_2 = x_2^{11} \qquad \lambda X_{15} = x_1 x_2^7 x_3^3 \qquad \lambda X_{28} = x_1 x_2^3 x_3^6 x_4$$

$$\lambda X_3 = x_3^{11} \qquad \lambda X_{16} = x_2^7 x_3^3 x_4 \qquad \lambda X_{29} = x_1^5 x_4^6$$

$$\lambda X_4 = x_4^{11} \qquad \lambda X_{17} = x_1^4 x_4^7 \qquad \lambda X_{30} = x_1^2 x_2^2 x_3^2 x_4^6$$

$$\lambda X_5 = x_1^{10} x_4 \qquad \lambda X_{18} = x_1 x_2 x_3^2 x_4^7 \qquad \lambda X_{31} = x_1^5 x_2^4 x_3^4$$

$$\lambda X_6 = x_1 x_4^{10} \qquad \lambda X_{19} = x_1^6 x_2^2 x_3^2 x_4^2 \qquad \lambda X_{32} = x_1^5 x_2 x_3^2 x_4$$

$$\lambda X_7 = x_1^9 x_4^2 \qquad \lambda X_{20} = x_1^6 x_4^5 \qquad \lambda X_{33} = x_2^2 x_3^4 x_4^5$$

$$\lambda X_8 = x_1^2 x_4^9 \qquad \lambda X_{21} = x_1^4 x_2^6 x_3 \qquad \lambda X_{34} = x_1^4 x_2^2 x_3^4 x_4$$

$$\lambda X_9 = x_1^8 x_4^3 \qquad \lambda X_{22} = x_1^3 x_2^6 x_3 x_4 \qquad \lambda X_{35} = x_1^4 x_2 x_3^2 x_4^4$$

$$\lambda X_{10} = x_1^8 x_2 x_3^2 \qquad \lambda X_{23} = x_2^6 x_3 x_4^4 \qquad \lambda X_{36} = x_1 x_2^2 x_3^4 x_4^4$$

$$\lambda X_{11} = x_1^3 x_4^8 \qquad \lambda X_{24} = x_1 x_2^6 x_3 x_4^3 \qquad \lambda X_{37} = x_1^2 x_2^2 x_3^4 x_4^3$$

$$\lambda X_{12} = x_2 x_3^2 x_4^8 \qquad \lambda X_{25} = x_1^2 x_2^6 x_3 x_4^2 \qquad \lambda X_{38} = x_1^3 x_2^2 x_3^4 x_4^2$$

$$\lambda X_{13} = x_1^7 x_2 x_3^2 x_4 \qquad \lambda X_{26} = x_1^2 x_2^3 x_3^6 \qquad \lambda X_{39} = x_1^3 x_2^2 x_3^2 x_4^5.$$

By eliminating λ, x_1, x_2, x_3, and x_4 from these 39 equations and $F_4(x_1, x_2, x_3, x_4) = 0$, we get as the 36 equations defining the surface:

$$\begin{vmatrix} X_1 & X_{11} & X_{13} & X_{15} & X_{19} & X_{30} & X_{32} & X_{33} & X_{34} & X_{36} & X_{37} & X_{38} \\ X_{10} & X_{12} & X_5 & X_{21} & X_7 & X_{29} & X_9 & X_{39} & X_{13} & X_{35} & X_{32} & X_{19} \end{vmatrix}$$

$$\begin{vmatrix} X_1 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{11} & X_{14} & X_{17} & X_{20} & X_{29} \\ X_5 & X_7 & X_4 & X_9 & X_6 & X_{14} & X_8 & X_{20} & X_{11} & X_{29} & X_{17} \end{vmatrix}$$

$$\begin{vmatrix} X_{26} & X_{28} & X_{31} & X_{34} & X_{36} & X_{38} \\ X_{32} & X_{35} & X_9 & X_{14} & X_{17} & X_{20} \end{vmatrix}$$

$$\begin{vmatrix} X_8 & X_{11} & X_{18} & X_{30} & X_{37} \\ X_{29} & X_{20} & X_{35} & X_{32} & X_{31} \end{vmatrix}$$

$$\begin{vmatrix} X_2 & X_{22} & X_{24} & X_{25} \\ X_{22} & X_{19} & X_{35} & X_{32} \end{vmatrix}$$

$$\begin{vmatrix} X_3 & X_{27} \\ X_{27} & X_{23} \end{vmatrix}$$

$$\begin{vmatrix} X_2 & X_{16} \\ X_{16} & X_{27} \end{vmatrix} \quad \text{and}$$

$$aX_{18} + bX_8 + cX_{11} + dX_4 = 0.$$

Designate by P_1' the branch point of ϕ corresponding to the point P_1 on F_4. The coordinates of P_1' are all zero except X_1.

The curves A_1 on F_4 pass through P_1 if $a_1 = 0$. The tangent plane at P_1 to F_4 is $x_4 = 0$. Now, the system of eleventh degree surfaces passing through P_1 cuts $x_4 = 0$ in the curves $x_4 = 0$,

$$a_2x_2^{11} + a_3x_3^{11} + a_{10}x_1^8x_2x_3^2 + a_{15}x_1^7x_2^3x_3 + a_{21}x_1^4x_2^6x_3$$

$$+ a_{26}x_1^2x_2^3x_3^6 + a_{31}x_1^5x_2^2x_3^4 = 0.$$

For general values of the constants this is an eleventh degree curve with a triple point at P_1, two branches being tangent to the line $x_3 = x_4 = 0$ and one to the line $x_2 = x_4 = 0$. When $a_{10} = a_{15} = a_{21} = a_{26} = a_{31} = 0$, the plane eleventh degree curve breaks up into eleven lines through P_1. These are all distinct except when either $a_2 = 0$ or $a_3 = 0$. Thus they coincide when $x_3 = x_4 = 0$ or $x_2 = x_4 = 0$, respectively. Since P_1 is imperfect, the $|A_1|$ curves thru P_1 must have eleven distinct branches unless each branch touches one of the two invariant directions. In the plane $x_4 = 0$, the involution I_{11} is generated by the homography T_1 which is

$$x_1' : x_2' : x_3' = x_1 : Ex_2 : E^5x_3.$$

By use of the plane quadratic transformation

$$X, \qquad y_1 : y_2 : y_3 = w_1^2 : w_2w_3 : w_1w_2$$

$$X^{-1}, \qquad w_1 : w_2 : w_3 = y_1y_3 : y_3^2 : y_1y_2$$

we consider the characteristics of the point P_1 along the invariant direction $x_2 = x_4 = 0$.

$$(w_1, w_2, w_3) \xrightarrow{X^{-1}} (y_1y_3, y_3^2, y_1y_2) \xrightarrow{T_1}$$

$$(E^5y_1y_3, E^{10}y_3^2, Ey_1y_2) \xrightarrow{X} (E^4w_1, E^9w_2, w_3) \quad \text{or}$$

$$x_1' : x_2' : x_3' = E^4x_1 : E^9x_2 : x_3 \qquad \text{for } T_2.$$

$$(w_1, w_2, w_3) \xrightarrow{X^{-1}} (y_1y_3, y_3^2, y_1y_2) \xrightarrow{T_2}$$

$$(E^4y_1y_3, y_3^2, E^2y_1y_2) \xrightarrow{X} (E^4w_1, w_2, E^2w_3) \quad \text{or}$$

$$x_1' : x_2' : x_3' = E^4x_1 : x_2 : E^2x_3 \qquad \text{for } T_3.$$

$$(w_1, w_2, w_3) \xrightarrow{X^{-1}} (y_1y_3, y_3^2, y_1y_2) \xrightarrow{T_3}$$

$$(E^6y_1y_3, E^4y_3^2, E^4y_1y_2) \xrightarrow{X} (E^2w_1, w_2, w_3). \quad \text{Our results}$$

are stated in the following theorem.

THEOREM 4. The imperfect point of coincidence has an imperfect point in the first and second order neighborhoods along the $x_2 = x_4 = 0$ direction. In the third order neighborhood there is a perfect point.

If we investigate the points adjacent to P_1 along $x_3 = x_4 = 0$, using the quadratic transformations

$$Y, \quad y_1 : y_2 : y_3 = w_1^2 : w_1w_2 : w_2w_3$$

$$Y^{-1}, \quad w_1 : w_2 : w_3 = y_1y_2 : y_2^2 : y_1y_3 \qquad \text{and}$$

$$Z, \quad y_1 : y_2 : y_3 = w_1^2 : w_2w_3 : w_1w_3$$

$$Z^{-1}, \quad w_1 : w_2 : w_3 = y_1y_3 : y_1y_2 : y_3^2 \qquad \text{one gets}$$

$$(w_1, w_2, w_3) \xrightarrow{Y^{-1}} (y_1y_2, y_2^2, y_1y_3) \xrightarrow{T_1}$$

$$(Ey_1y_2, E^2y_2^2, E^5y_1y_3) \xrightarrow{Y} (w_1, Ew_2, E^4w_3) \qquad \text{or}$$

$$x_1' : x_2' : x_3' = x_1 : Ex_2 : E^4x_3 \qquad \text{for } T_2.$$

$$\text{Again, } (w_1, w_2, w_3) \xrightarrow{Z^{-1}} (y_1y_3, y_1y_2, y_3^2) \xrightarrow{T_2}$$

$$(E^4y_1y_3, Ey_1y_2, E^8y_3^2) \xrightarrow{Z} (E^3w_1, w_2, E^7w_3) \qquad \text{or}$$

$$x_1' : x_2' : x_3' = E^3x_1 : x_2 : E^7x_3 \qquad \text{for } T_3.$$

$$\text{Finally, } (w_1, w_2, w_3) \xrightarrow{Z^{-1}} (y_1y_3, y_1y_2, y_3^2) \xrightarrow{T_3}$$

$$(E^{10}y_1y_3, E^3y_1y_2, E^3y_3^2) \xrightarrow{Z} (E^7w_1, w_2, w_3) .$$

Thus the following theorem has been established.

THEOREM 5. Along the invariant direction $x_3 = x_4 = 0$, there are no perfect points in the first or second order neighborhoods. There is, however, a perfect point in the third order neighborhood.

THEOREM 6. On the surface F_4, at point P_1, there are two perfect points in the tangent plane. Each is in the third order neighborhood of P_1.

CHAPTER V

MAPPING OF CERTAIN PLANE ELEVENTH DEGREE CURVES INVARIANT UNDER I_{11}

In this chapter, we shall consider the results of mapping a system of eleventh order plane algebraic curves, invariant under the cyclic involution

$$T, \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^5x_3, \qquad E^{11} = 1$$

onto a linear space S_7, of seven dimensions.

The homography T has three invariant points. They are $O_1(1,0,0)$, $O_2(0,1,0)$, and $O_3(0,0,1)$.

The general system of plane curves of order eleven is

$$a_1x_1^{11} + a_2x_1^8x_2x_3^2 + a_3x_1^5x_2^2x_3^4 + a_4x_1^4x_2^6x_3 + a_5x_1^2x_2^5x_3^4 + a_6x_1x_2^7x_3^3$$

$$+ a_7x_2^{11} + a_8x_3^{11} + a_9x_1^{10}x_2 + a_{10}x_1^7x_2^2x_3 + a_{11}x_1^4x_2^5x_3 + a_{12}x_1^3x_2^7x_3$$

$$+ a_{13}x_1^2x_3^9 + a_{14}x_1x_2^4x_3^6 + a_{15}x_2^8x_3^3 + a_{16}x_1^9x_2^2 + a_{17}x_1^6x_2^3x_3^2 + a_{18}x_1^4x_3^7$$

$$+ a_{19}x_1^3x_2^4x_3^4 + a_{20}x_1^2x_2^8x_3 + a_{21}x_1x_2x_3^9 + a_{22}x_2^5x_3^6 + a_{23}x_1^8x_2^3 + a_{24}x_1^6x_3^5$$

$$+ a_{25}x_1^5x_2^4x_3^2 + a_{26}x_1^3x_2x_3^7 + a_{27}x_1^2x_2^5x_3^4 + a_{28}x_1x_2^9x_3 + a_{29}x_2^2x_3^9$$

$$+ a_{30}x_1^8x_3^3 + a_{31}x_1^7x_2^4 + a_{32}x_1^5x_2x_3^5 + a_{33}x_1^4x_2^5x_3^2 + a_{34}x_1^2x_2^2x_3^7$$

$$+ a_{35}x_1^6x_2^4x_3 + a_{36}x_2^{10}x_3 + a_{37}x_1^{10}x_3 + a_{38}x_1^7x_2^3x_3 + a_{39}x_1^6x_2^5$$

$$+ a_{40}x_1^4x_2^2x_3^5 + a_{41}x_1^3x_2^6x_3^2 + a_{42}x_1x_2^3x_3^7 + a_{43}x_2^7x_3^4 + a_{44}x_1^9x_2x_3$$

$$+ a_{45}x_1^6x_2^2x_3^3 + a_{46}x_1^5x_2^6 + a_{47}x_1^5x_2^3x_3^5 + a_{48}x_1^2x_2^7x_3^2 + a_{49}x_1x_3^{10}$$

$$+ a_{50}x_2^4x_3^7 + a_{51}x_1^8x_2^2x_3 + a_{52}x_1^5x_2^3x_3^3 + a_{53}x_1^4x_2^7 + a_{54}x_1^5x_3^8$$

$$+ a_{55}x_1^2x_2^4x_3^5 + a_{56}x_1^8x_2^2x_3 + a_{57}x_2^{10}x_3 + a_{58}x_1^7x_2^3x_3 + a_{59}x_1^5x_3^6$$

$$+ a_{60}x_1^4x_2^4x_3^3 + a_{61}x_1^3x_2^8 + a_{62}x_1^2x_2^8x_3 + a_{63}x_1^5x_2^5x_3 + a_{64}x_1^9x_2^2x_3$$

$$+ a_{65}x_1^7x_3^4 + a_{66}x_1^6x_2^4x_3 + a_{67}x_1^4x_2^6x_3 + a_{68}x_1^3x_2^5x_3^3 + a_{69}x_1^2x_2^9$$

$$+ a_{70}x_1x_2^2x_3^8 + a_{71}x_1^6x_3^5 + a_{72}x_1^9x_3^2 + a_{73}x_1^6x_2^4x_3 + a_{74}x_1^5x_2^5x_3$$

$$+ a_{75}x_1^3x_2^2x_3^6 + a_{76}x_1^2x_2^6x_3^3 + a_{77}x_1x_2^{10} + a_{78}x_2^3x_3^8 = 0.$$

It is, in general, non-invariant under the transformation T.

This equation can be split up, however, into eleven equations, which are each invariant. They are

$|C_0|$, $a_1x_1^{11} + a_2x_1^8x_2x_3^2 + a_3x_1^5x_2^2x_3^4 + a_4x_1^4x_2^4x_3^6 + a_5x_1^2x_2^3x_3^6 + a_6x_1x_2^7x_3^3$
$+ a_7x_2^{11} + a_8x_3^{11} = 0.$

$|C_1|$, $b_1x_1^{10}x_2 + b_2x_1^7x_2^2x_3^2 + b_3x_1^4x_2^3x_3^4 + b_4x_1^3x_2^7x_3 + b_5x_1^2x_3^9 + b_6x_1x_2^4x_3^6$
$+ b_7x_2^8x_3^3 = 0.$

$$|c_2|, \quad c_1x_1^9x_2^2 + c_2x_1^6x_2^3x_3^2 + c_3x_1^4x_3^7 + c_4x_1^3x_2^4x_3^4 + c_5x_1^2x_2^8x_3 + c_6x_1x_2^9x_3^5$$
$$+ c_7x_2^5x_3^6 = 0.$$

$$|c_3|, \quad d_1x_1^8x_2^3 + d_2x_1^6x_3^5 + d_3x_1^5x_2^4x_3^2 + d_4x_1^3x_2x_3^7 + d_5x_1^2x_2^5x_3^4 + d_6x_1x_2^9x_3$$
$$+ d_7x_2^2x_3^9 = 0.$$

$$|c_4|, \quad e_1x_1^8x_3^3 + e_2x_1^7x_2^4 + e_3x_1^5x_2x_3^5 + e_4x_1^4x_2^5x_3^2 + e_5x_1^2x_2^2x_3^7 + e_6x_1x_2^6x_3^4$$
$$+ e_7x_2^{10}x_3 = 0.$$

$$|c_5|, \quad f_1x_1^{10}x_3 + f_2x_1^7x_2x_3^3 + f_3x_1^6x_2^5 + f_4x_1^4x_2^2x_3^5 + f_5x_1^3x_2^6x_3^2 + f_6x_1x_2^3x_3^7$$
$$+ f_7x_2^7x_3^4 = 0.$$

$$|c_6|, \quad g_1x_1^9x_2x_3 + g_2x_1^6x_2^2x_3^3 + g_3x_1^5x_2^6 + g_4x_1^3x_2^3x_3^5 + g_5x_1^2x_2^7x_3^2 + g_6x_1x_3^{10}$$
$$+ g_7x_2^4x_3^7 = 0.$$

$$|c_7|, \quad h_1x_1^8x_2^2x_3 + h_2x_1^5x_2^3x_3^3 + h_3x_1^4x_2^7 + h_4x_1^3x_3^8 + h_5x_1^2x_2^4x_3^5 + h_6x_1x_2^8x_3^2$$
$$+ h_7x_2x_3^{10} = 0.$$

$$|c_8|, \quad j_1x_1^7x_2^3x_3 + j_2x_1^5x_3^6 + j_3x_1^4x_2^4x_3^3 + j_4x_1^3x_2^8 + j_5x_1^2x_2x_3^8 + j_6x_1x_2^5x_3^5$$
$$+ j_7x_2^9x_3^2 = 0.$$

$$|c_9|, \quad k_1x_1^7x_3^4 + k_2x_1^6x_2^4x_3 + k_3x_1^4x_2x_3^6 + k_4x_1^3x_2^5x_3^3 + k_5x_1^2x_2^9 + k_6x_1x_2^8x_3^3$$
$$+ k_7x_2^6x_3^5 = 0.$$

$$|C_{10}|, \quad l_1x_1^9x_3^2 + l_2x_1^6x_2x_3^4 + l_3x_1^5x_2^5x_3 + l_4x_1^3x_2^2x_3^6 + l_5x_1^2x_2^6x_3^3 + l_6x_1x_2^{10}$$
$$+ l_7x_2^3x_3^8 = 0.$$

Refer now the curves $|C_0|$ to a linear space S_7 of seven dimensions using the transformation

$$(2) \qquad \frac{X_1}{x_1^{11}} = \frac{X_2}{x_1^8x_2x_3^2} = \frac{X_3}{x_1^5x_2^2x_3^4} = \frac{X_4}{x_1^4x_2^6x_3} = \frac{X_5}{x_1^2x_2^3x_3^6} = \frac{X_6}{x_1x_2^7x_3^3} = \frac{X_7}{x_2^{11}} = \frac{X_8}{x_3^{11}}.$$

By eliminating the x_i's from these equations, one gets for the new surface F the equations

$$(3) \qquad \left\| \begin{matrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 \\ X_2 & X_3 & X_5 & X_6 & X_6X_8 & X_5X_7 \end{matrix} \right\| = 0,$$

or

(a) $X_1X_3 = X_2^2$

(b) $X_2X_5 = X_3^2$

(c) $X_3X_6 = X_4X_5$

(d) $X_4X_8 = X_5^2$

(e) $X_5X_7 = X_6^2$.

These five equations are independent. From them five dependent equations are obtained also. These help one in determining the order of surface F. They are

(f) $X_1X_5 = X_2X_3$

(g) $X_1X_6 = X_2X_4$

(h) $X_2X_6 = X_3X_4$

(i) $X_3X_6X_8 = X_5^3$

(j) $X_5X_7 = X_4X_6$.

Order of a Surface

The order of surface F may be obtained by using the method of F. S. Woods [19], which consists of determining the number of points of intersection the surface makes with two general cutting hyperplanes. Thus, if the surface is represented by N - 2 equations in a space of N dimensions, it is necessary to solve N equations simultaneously. In this case the equations to be solved are

1. $X_1X_3 = X_2^2$

2. $X_2X_5 = X_3^2$

3. $X_3X_6 = X_4X_5$

4. $X_4X_8 = X_5^2$

5. $X_5X_7 = X_6^2$

6. $a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6 + a_7X_7 + a_8X_8 = 0$

7. $b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4 + b_5X_5 + b_6X_6 + b_7X_7 + b_8X_8 = 0.$

By elimination of variables, the seven equations are reduced to one equation in one variable of degree 11.

Image Curves

We shall designate by $|\Sigma_0|$ the hyperplane sections of F, which correspond to the curves $|C_0|$. Likewise, $|\Sigma_1|$, $|\Sigma_2|$, . . . , $|\Sigma_{10}|$ are the curves on F which correspond, respectively, to the curves $|C_1|$, $|C_2|$, . . . , $|C_{10}|$.

The equations of these curves are

$$|\Sigma_1|, \quad \left|\left| \begin{matrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 & b_1X_1+b_2X_2+b_3X_3+b_4X_4+b_6X_5+b_7X_6 \\ X_2 & X_3 & X_5 & X_6 & X_6X_8 & X_5X_7 & -b_5X_8 \end{matrix} \right|\right|$$

$$|\Sigma_2|, \quad \left|\left| \begin{matrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 & -b_1X_1X_5-b_4X_4X_8 \\ X_2 & X_3 & X_5 & X_6 & X_6X_8 & X_5X_7 & b_2X_3X_5+b_3X_3X_8+b_5X_5X_6+b_6X_5X_8+b_7X_6X_8 \end{matrix} \right|\right|$$

$$|\Sigma_3|, \quad \left|\left| \begin{matrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 & b_2X_1X_5+b_3X_2X_4+b_4X_5^2+b_5X_3X_4+b_6X_2X_7 \\ X_2 & X_3 & X_5 & X_6 & X_6X_8 & X_5X_7 & -b_1X_2X_4-b_7X_4X_8 \end{matrix} \right|\right|$$

$$|\Sigma_4|, \quad \left|\left| \begin{matrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 & -b_4X_3^2 \ -b_5X_2X_8 \\ X_2 & X_3 & X_5 & X_6 & X_6X_8 & X_5X_7 & b_1X_1X_8+b_2X_2X_5+b_3X_2X_8+b_6X_5^2+b_7X_5X_6 \end{matrix} \right|\right|$$

$$|\Sigma_5|, \quad \begin{vmatrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 & b_2X_2+b_4X_3+b_6X_5+b_7X_6 \\ X_2 & X_3 & X_5 & X_6 & X_6X_8 & X_5X_7 & -b_1X_2-b_3X_4-b_5X_6 \end{vmatrix}$$

$$|\Sigma_6|, \quad \begin{vmatrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 & -b_7X_5 \\ X_2 & X_3 & X_5 & X_6 & X_6X_8 & X_5X_7 & b_1X_2+b_2X_3+b_3X_4+b_4X_5+b_5X_6+b_6X_8 \end{vmatrix}$$

$$|\Sigma_7|, \quad \begin{vmatrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 & -b_4X_2X_3-b_7X_2X_5 \\ X_2 & X_5 & X_5 & X_6 & X_6X_8 & X_5X_7 & b_1X_1X_4+b_2X_2X_4+b_3X_1X_7+b_5X_3X_4+b_6X_2X_7 \end{vmatrix}$$

$$|\Sigma_8|, \quad \begin{vmatrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 & -b_5X_5-b_6X_6-b_7X_7 \\ X_2 & X_3 & X_5 & X_6 & X_6X_8 & X_5X_7 & b_1X_4+b_2X_5+b_3X_6+b_4X_7 \end{vmatrix}$$

$$|\Sigma_9|, \quad \begin{vmatrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 & -b_6X_5-b_7X_6 \\ X_2 & X_3 & X_5 & X_6 & X_6X_8 & X_5X_7 & b_1X_3+b_2X_4+b_3X_5+b_4X_6+b_5X_7 \end{vmatrix}$$

$$|\Sigma_{10}|, \quad \begin{vmatrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 & -b_7X_5 \\ X_2 & X_3 & X_5 & X_6 & X_6X_8 & X_5X_7 & b_1X_2+b_2X_3+b_3X_4+b_4X_5+b_5X_6+b_6X_7 \end{vmatrix}.$$

CHAPTER VI

A STUDY OF BRANCH POINT O_1'

Branch Point O_1'

To point $O_1(1,0,0)$ on the plane corresponds on F the point $O_1'(1,0,0,0,0,0,0)$. The singularities of this point will now be studied. Let us consider the family (∞^6) of curves from the system $|C_0|$, passing thru the invariant point O_1. The equation of this family of curves is

$$(4)\qquad a_2x_1^8x_2x_3^2 + a_3x_1^5x_2^4x_3 + a_4x_1^4x_2^6x_3 + a_5x_1^2x_2^3x_3^6 + a_6x_1x_2^7x_3^3 + a_7x_2^{11} + a_8x_3^{11} = 0.$$

Consider the transformation

$$U,\qquad x_1 : x_2 : x_3 = z_1^2 : z_1z_2 : z_2z_3$$

Applying the transformation U four times to equation (4), we get

$$[A]\qquad a_2z_1^{17}z_3^2 + a_3z_1^{12}z_2^3z_3^4 + a_4z_1^{14}z_2^4z_3 + a_5z_1^7z_2^6z_3^6 + a_6z_1^9z_2^7z_3^3 + a_7z_1^{11}z_2^8 + a_8z_2^8z_3^{11} = 0.$$

$$[B]\qquad a_2z_1^{26}z_3^2 + a_3z_1^{19}z_2^5z_3^4 + a_4z_1^{24}z_2^3z_3 + a_5z_1^{12}z_2^{10}z_3^6 + a_6z_1^{17}z_2^8z_3^3 + a_7z_1^{22}z_2^6 + a_8z_2^{17}z_3^{11} = 0.$$

$$[C]\qquad a_2z_1^{35}z_3^2 + a_3z_1^{26}z_2^7z_3^4 + a_4z_1^{34}z_2^2z_3 + a_5z_1^{17}z_2^{14}z_3^6 + a_6z_1^{25}z_2^9z_3^3 + a_7z_1^{33}z_2^4 + a_8z_2^{26}z_3^{11} = 0.$$

$$(5)\qquad z_1^{44}(a_2z_3^2+a_4z_2z_3+a_7z_2^2) + a_3z_1^{33}z_2^9z_3^4 + a_5z_1^{22}z_2^{18}z_3^6 + a_6z_1^{33}z_2^{10}z_3^3 + a_8z_2^{35}z_3^{11} = 0.$$

This shows that the double point $(z_2 = z_3 = 0)$, corresponds to the point 0_{12222} in the fourth order neighborhood of 0_1 in the $x_5 = 0$ direction.

In order to obtain the points, infinitely near $0_1'(1,0,0,0,0,0,0,0)$ on F, which correspond to the points infinitely near the point $(z_2 = z_3 = 0)$, it is necessary, first, to project the surface F from the point $0_1'$ onto the hyperplane $X_1 = 0$. This gives the independent equations

F_1,

$$X_4X_8 = X_5^2$$

$$X_2X_5 = X_3^2$$

$$X_3X_6 = X_4X_5$$

$$X_5X_7 = X_6^2 \quad \text{and } X_1 = 0$$

and the dependent equations

$$X_3X_6X_8 = X_5^3$$

$$X_2X_6 = X_3X_4$$

$$X_3X_7 = X_4X_6$$

The order of this surface is 8, found by Coble's Method.†

Now, apply the transformation U four times to

---

†See Appendix I.

$$\frac{X_2}{x_1^8 x_2^2} = \frac{X_3}{x_1^5 x_2^2 x_3^4} = \frac{X_4}{x_1^4 x_2^6 x_3} = \frac{X_5}{x_1^2 x_2^3 x_3^6} = \frac{X_6}{x_1 x_2^7 x_3^3} = \frac{X_7}{x_2^{11}} = \frac{X_8}{x_3^{11}}$$

and obtain the simplified expression

$$\frac{X_2}{z_1^{44} z_3^2} = \frac{X_3}{z_1^{33} z_2^9 z_3^4} = \frac{X_4}{z_1^{44} z_2 z_3} = \frac{X_5}{z_1^{22} z_2^{18} z_3^6} = \frac{X_6}{z_1^{33} z_2^{10} z_3^3} = \frac{X_7}{z_1^{44} z_2^2} = \frac{X_8}{z_2^{35} z_3^{11}}.$$

Since we are interested in approaching the point $(z_2 = z_3 = 0)$ from all directions, let $z_3 = kz_2$ and substitute in the last equations. This gives us

$$\frac{X_2}{z_1^{44} k^2 z_2^2} = \frac{X_3}{z_1^{33} k^4 z_2^{13}} = \frac{X_4}{z_1^{44} k z_2^2} = \frac{X_5}{z_1^{22} k^6 z_2^{24}} = \frac{X_6}{z_1^{33} k^3 z_2^{13}} = \frac{X_7}{z_1^{44} z_2^2} = \frac{X_8}{k^{11} z_2^{46}}$$

which simplifies to

$$\frac{X_2}{k^2 z_1^{44}} = \frac{X_3}{k^4 z_1^{33} z_2^{11}} = \frac{X_4}{k z_1^{44}} = \frac{X_5}{k^6 z_1^{22} z_2^{22}} = \frac{X_6}{k^3 z_1^{33} z_2^{11}} = \frac{X_7}{z_1^{44}} = \frac{X_8}{k^{11} z_2^{44}}.$$

Passing to the limit as z_2 goes to 0, the equations above will have meaning if $X_3 = X_5 = X_6 = X_8 = 0$. This gives

$$\frac{X_2}{k^2 z_1^{44}} = \frac{X_4}{k z_1^{44}} = \frac{X_7}{z_1^{44}}.$$

Multiplying thru by z_1^{44}, we get

$$\frac{X_2}{k^2} = \frac{X_4}{k} = \frac{X_7}{1}, \quad X_3 = X_5 = X_6 = X_8 = 0. \tag{6}$$

It follows that

$$\frac{X_2}{X_4} = k \qquad \frac{X_4}{X_7} = k, \qquad X_3 = X_5 = X_6 = X_8 = 0.$$

Eliminating k, one gets

$$X_2X_7 = X_4^2$$
$$X_3 = X_5 = X_6 = X_8 = 0.$$

In S_7, we have a quadric cone,

$$X_2X_7 = X_4^2, \quad X_3 = X_5 = X_6 = X_8 = 0,$$

which is tangent to F at $0_1'$. It intersects F_1 in

$$(G_1) \qquad X_2X_7 = X_4^2, \qquad X_1 = X_3 = X_5 = X_6 = X_8 = 0$$

which is a conic in S_7 on the tangent cone. This shows that the points in the first order neighborhood of the point $(z_2 = z_3 = 0)$, as well as the points in the first order neighborhood of 0_{12222}, correspond projectively.

When we apply the quadratic transformation

$$V, \qquad x_1 : x_2 : x_3 = z_1^2 : z_2z_3 : z_1z_3$$

eight times to the curves (4) we get

$$[A] \qquad a_2z_1^{18}z_2 + a_3z_1^{14}z_2^2z_3^3 + a_4z_1^9z_2^6z_3^4 + a_5z_1^{10}z_2^3z_3^6 + a_6z_1^5z_2^7z_3^7 + a_7z_2^{11}z_3^8 + a_8z_1^{11}z_3^8 = 0.$$

$$[B] \qquad a_2z_1^{28}z_2 + a_3z_1^{23}z_2^2z_3^4 + a_4z_1^{14}z_2^6z_3^9 + a_5z_1^{18}z_2^3z_3^8 + a_6z_1^9z_2^7z_3^{13} + a_7z_2^{11}z_3^{18} + a_8z_1^{22}z_3^7 = 0.$$

$$[C] \quad a_2z_1^{38}z_2 + a_3z_1^{32}z_2^2z_3^5 + a_4z_1^{19}z_2^6z_3^{14} + a_5z_1^{26}z_2^3z_3^{10} + a_6z_1^{13}z_2^7z_3^{19} + a_7z_2^{11}z_3^{28} + a_8z_1^{33}z_3^6 = 0.$$

$$[D] \quad a_2z_1^{48}z_2 + a_3z_1^{41}z_2^2z_3^6 + a_4z_1^{24}z_2^6z_3^{19} + a_5z_1^{34}z_2^3z_3^{12} + a_6z_1^{17}z_2^7z_3^{25} + a_7z_2^{11}z_3^{38} + a_8z_1^{44}z_3^5 = 0.$$

$$[E] \quad a_2z_1^{58}z_2 + a_3z_1^{50}z_2^2z_3^7 + a_4z_1^{29}z_2^6z_3^{24} + a_5z_1^{42}z_2^3z_3^{14} + a_6z_1^{21}z_2^7z_3^{31} + a_7z_2^{11}z_3^{48} + a_8z_1^{55}z_3^4 = 0.$$

$$[F] \quad a_2z_1^{68}z_2 + a_3z_1^{59}z_2^2z_3^8 + a_4z_1^{34}z_2^6z_3^{29} + a_5z_1^{50}z_2^3z_3^{16} + a_6z_1^{25}z_2^7z_3^{37} + a_7z_2^{11}z_3^{58} + a_8z_1^{66}z_3^3 = 0.$$

$$[G] \quad a_2z_1^{78}z_2 + a_3z_1^{68}z_2^2z_3^9 + a_4z_1^{39}z_2^6z_3^{34} + a_5z_1^{58}z_2^3z_3^{18} + a_6z_1^{29}z_2^7z_3^{43} + a_7z_2^{11}z_3^{68} + a_8z_1^{77}z_3^2 = 0.$$

$$(7) \quad z_1^{88}(a_2z_2+a_8z_3) + a_3z_1^{77}z_2^2z_3^{10} + a_4z_1^{44}z_2^6z_3^{39} + a_5z_1^{66}z_2^3z_3^{20} + a_6z_1^{33}z_2^7z_3^{49} + a_7z_2^{11}z_3^{78} = 0.$$

This shows that to $0_{133333333}$ in the eighth order neighborhood of 0_1 on $x_2 = 0$ corresponds simply the point $(z_2 = z_3 = 0)$. In referring to a point such as $0_{133333333}$, we shall designate it as $0_{13(8)}$. In general $0_{ab(c)}$ signifies that a is followed by a number of b's equal to the integer c.

Applying the quadratic transformation V eight times is equivalent to using the transformation V_8,

$$x_1 : x_2 : x_3 = z_1^9 : z_2z_3^8 : z_1^8z_3 \text{ once.}$$

Use this transformation on the curves of the x-plane and then project the resultant curves of the z-plane over onto S_7, by means of

$$\frac{X_2}{x_1^8 x_2 x_3^2} = \frac{X_3}{x_1^5 x_2^2 x_3^4} = \frac{X_4}{x_1^4 x_2^6 x_3} = \frac{X_5}{x_1^2 x_2^3 x_3^6} = \frac{X_6}{x_1 x_2^7 x_3^3} = \frac{X_7}{x_2^{11}} = \frac{X_8}{x_3^{11}}$$

and

$$\frac{X_2}{z_1^{88} z_2} = \frac{X_3}{z_1^{77} z_2^2 z_3^{10}} = \frac{X_4}{z_1^{44} z_2^6 z_3^{39}} = \frac{X_5}{z_1^{66} z_2^3 z_3^{20}} = \frac{X_6}{z_1^{33} z_2^7 z_3^{49}} = \frac{X_7}{z_2^{11} z_3^{78}} = \frac{X_8}{z_1^{88} z_3}.$$

Again let $z_3 = kz_2$ and substitute. One gets

$$\frac{X_2}{z_1^{88} z_2} = \frac{X_3}{k^{10} z_1^{77} z_2^{12}} = \frac{X_4}{k^{39} z_1^{44} z_2^{45}} = \frac{X_5}{k^{20} z_1^{66} z_2^{23}} = \frac{X_6}{k^{49} z_1^{33} z_2^{56}} = \frac{X_7}{k^{78} z_2^{89}} = \frac{X_8}{k z_1^{88} z_2}$$

or

$$\frac{X_2}{z_1^{88}} = \frac{X_3}{k^{10} z_1^{77} z_2^{11}} = \frac{X_4}{k^{39} z_1^{44} z_2^{44}} = \frac{X_5}{k^{20} z_1^{66} z_2^{22}} = \frac{X_6}{k^{49} z_1^{33} z_2^{55}} = \frac{X_7}{k^{78} z_2^{88}} = \frac{X_8}{k z_1^{88}}.$$

When allowing z_2 to approach zero, this reduces to

(8) $$X_8 = kX_2, \quad X_3 = X_4 = X_5 = X_6 = X_7 = 0.$$

It follows, therefore, that to the points of the straight line $X_1 = X_3 = X_4 = X_5 = X_6 = X_7 = 0$, denoted by (a_1), correspond the points infinitely near the point $0_{133333333}$, denoted by $0_{13(8)}$.

This also means that the points infinitely near $0_{13(8)}$ correspond to the points infinitely near the point $0_1'$ on F and lying in the plane

$$X_3 = X_4 = X_5 = X_6 = X_7 = 0.$$

This plane is tangent to F at the point $0_1'$.

It has now been established that to the invariant point O_1 of the involution I_{11}, corresponds on the surface F a branch point O_1', which is a 3-tuple point. The curve (C_1) and the line (a_1) have in common only the point $X_1 = X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0$ designated by $O_2'(0,1,0,0,0,0,0,0)$.

Furthermore, b_1, the line

$$X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0$$

lies in the tangent plane as well as on the quadric cone. Thus, the tangent plane is tangent to the quadric cone along line b_1.

Not only does the line a_1 and curve C_1 intersect at $O_2'(0,1,0,0,0,0,0,0)$, but b_1 also intersects the curve and line a_1 at this same point.

Images of Curves $|C_1|$ at O_1'

Consider now the study of the system of curves $|C_1|$,

$$b_1x_1^{10}x_2 + b_2x_1^7x_2^2x_3 + b_3x_1^4x_2^3x_3^4 + b_4x_1^3x_2^7x_3 + b_5x_1^2x_3^9 + b_6x_1^4x_2^6x_3 + b_7x_2^8x_3^3 = 0.$$

A member of the system $|C_1|$ passes thru the invariant point O_1, which is simple for this curve. This curve is tangent to the invariant direction $x_2 = 0$.

Applying the transformation V eight times to the curves $|C_1|$ gives

[A] $$b_1z_1^{17}z_2 + b_2z_1^{15}z_2^2z_3 + b_3z_1^9z_2^3z_3^6 + b_4z_1^4z_2^7z_3^7 + b_5z_1^{10}z_3^8 + b_6z_1^5z_2^4z_3^9 + b_7z_2^8z_3^{10} = 0.$$

[B] $$b_1 z_1^{24} z_2 + b_2 z_1^{19} z_2^2 z_3^4 + b_3 z_1^{14} z_2^3 z_3^8 + b_4 z_1^5 z_2^7 z_3^{13} + b_5 z_1^{18} z_3^7 + b_6 z_1^9 z_2^4 z_3^{12} + b_7 z_2^8 z_3^{17} = 0.$$

[C] $$b_1 z_1^{31} z_2 + b_2 z_1^{25} z_2^2 z_3^5 + b_3 z_1^{19} z_2^3 z_3^{10} + b_4 z_1^6 z_2^7 z_3^{19} + b_5 z_1^{26} z_3^6 + b_6 z_1^{13} z_2^4 z_3^{15} + b_7 z_2^8 z_3^{24} = 0.$$

[D] $$b_1 z_1^{38} z_2 + b_2 z_1^{31} z_2^2 z_3^6 + b_3 z_1^{24} z_2^3 z_3^{12} + b_4 z_1^7 z_2^7 z_3^{25} + b_5 z_1^{34} z_3^5 + b_6 z_1^{17} z_2^4 z_3^{18} + b_7 z_2^8 z_3^{31} = 0.$$

[E] $$b_1 z_1^{45} z_3 + b_2 z_1^{37} z_2^2 z_3^7 + b_3 z_1^{29} z_2^3 z_3^{14} + b_4 z_1^8 z_2^7 z_3^{31} + b_5 z_1^{42} z_3^4 + b_6 z_1^{21} z_2^4 z_3^{21} + b_7 z_2^8 z_3^{38} = 0.$$

[F] $$b_1 z_1^{52} z_2 + b_2 z_1^{43} z_2^2 z_3^8 + b_3 z_1^{34} z_2^3 z_3^{16} + b_4 z_1^9 z_2^7 z_3^{37} + b_5 z_1^{50} z_3^3 + b_6 z_1^{25} z_2^4 z_3^{24} + b_7 z_2^8 z_3^{45} = 0.$$

[G] $$b_1 z_1^{59} z_2 + b_2 z_1^{49} z_2^2 z_3^9 + b_3 z_1^{39} z_2^3 z_3^{18} + b_4 z_1^{10} z_2^7 z_3^{43} + b_5 z_1^{58} z_3^2 + b_6 z_1^{29} z_2^4 z_3^{27} + b_7 z_2^8 z_3^{52} = 0.$$

(9) $$z_1^{66}(b_1 z_2 + b_5 z_3) + b_2 z_1^{55} z_2^2 z_3^{10} + b_3 z_1^{44} z_2^3 z_3^{20} + b_4 z_1^{11} z_2^7 z_3^{49} + b_6 z_1^{33} z_2^4 z_3^{30} + b_7 z_2^8 z_3^{59} = 0.$$

We note that 0_{13}, 0_{133}, 0_{1333}, . . . , $0_{13(8)}$ are all simple points. Furthermore to point $0_{13(8)}$ in the eighth order neighborhood of 0_1 on $x_2 = 0$ corresponds simply the point $(z_2 = z_3 = 0)$. The tangent to this curve at this point has for its equation

$$b_1 z_2 + b_5 z_3 = 0.$$

To approach the point $(z_2 = z_3 = 0)$ along the curve (9), substitute $z_3 = kz_2$ into its equation and allow z_2 to approach zero. The results are

$$z_1^{66}(b_1 z_2 + b_5 k z_2) + b_2 k^{10} z_1^{55} z_2^{12} + b_3 k^{20} z_1^{44} z_2^{23} + b_4 k^{49} z_1^{11} z_2^{56} + b_6 k^{30} z_1^{33} z_2^{34} + b_7 k^{59} z_2^{67} = 0.$$

$$z_1^{66}(b_1 + b_5 k) + b_2 k z_1^{55} z_2^{11} + b_3 k^{20} z_1^{44} z_2^{22} + b_4 k^{49} z_1^{11} z_2^{55} + b_6 k^{30} z_1^{33} z_2^{33} + b_7 k^{59} z_2^{66} = 0,$$

and

$$b_1 + b_5 k = 0.$$

The value of k from this equation, substituted into equations (8), locates the point on the line a_1, corresponding to the point on $|C_1|$, in the first order neighborhood of $0_{13(8)}$. The coordinates of this point on a_1 are

$$(0, b_5, 0, 0, 0, 0, 0, -b_1).$$

The curve Σ_1 on F in S_7 passes thru the point $0_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and $0_1'$. This tangent line is represented by

$$X_3 = X_4 = X_5 = X_6 = X_7 = 0$$

$$b_1X_2 + b_5X_8 = 0.$$

Here O_1' is a simple point for the Σ_1 curve. It passes thru O_1' along a line in the tangent plane.

Images of $|C_2|$ at O_1'

One now investigates the system $|C_2|$. Its equation is

$$c_1x_1^9x_2^2 + c_2x_1^6x_2^3x_3^2 + c_3x_1^4x_3^7 + c_4x_1^3x_2^4x_3^4 + c_5x_1^2x_2^8x_3 + c_6x_1x_2^9x_3^5 + c_7x_2^5x_3^6 = 0.$$

It has a double point at O_1, with tangent $x_2 = 0$ counted twice.

Applying the transformation V to this equation 3 times, followed by the application of U once gives

[A] $$c_1z_1^{13}z_2^2 + c_2z_1^9z_2^3z_3^3 + c_3z_1^{10}z_3^5 + c_4z_1^5z_2^4z_3^6 + c_5z_2^8z_3^7 + c_6z_1^6z_2^2z_3^8 + c_7z_1z_2^5z_3^9 = 0.$$

[B] $$c_1z_1^{19}z_2^2 + c_2z_1^{14}z_2^3z_3^4 + c_3z_1^{18}z_3^3 + c_4z_1^9z_2^4z_3^8 + c_5z_2^8z_3^{13} + c_6z_1^{13}z_2z_3^7 + c_7z_1^4z_2^5z_3^{12} = 0.$$

[C] $$c_1z_1^{25}z_2^2 + c_2z_1^{19}z_2^3z_3^5 + c_3z_1^{26}z_3^3 + c_4z_1^{13}z_2^4z_3^{10} + c_5z_2^8z_3^{19} + c_6z_1^{20}z_2z_3^6 + c_7z_1^7z_2^5z_3^{15} = 0.$$

$$(10)\qquad z_1^{44}(c_1z_2+c_3z_3) + c_2z_1^{33}z_2^{7}z_3^{5} + c_4z_1^{22}z_2^{13}z_3^{10} + c_5z_2^{26}z_3^{19}$$

$$+ c_6z_1^{33}z_2^{6}z_3^{6} + c_7z_1^{11}z_2^{19}z_3^{15} = 0.$$

This shows that the curve C_2 passes doubly thru the points 0_{13}, 0_{133}, and 0_{1333} in the first, second, and third order neighborhoods of 0_1. It passes simply thru 0_{13332} on $x_2 = 0$.

To approach the point $(z_2 = z_3 = 0)$ along the curve (10), substitute $z_3 = kz_2$ into its equation and allow z_2 to approach zero. The results are

$$z_1^{44}(c_1z_2+c_3kz_2) + c_2k^5c_1^{33}c_2^{12} + c_4k^{10}c_1^{22}c_2^{23} + c_5k^{19}z_2^{45}$$

$$+ c_6k^6z_1^{33}z_2^{12} + c_7k^{15}z_1^{11}z_2^{34} = 0.$$

$$z_1^{44}(c_1+c_3k) + c_2k^5c_1^{33}c_2^{11} + c_4k^{10}c_1^{22}c_2^{22} + c_5k^{19}z_2^{44}$$

$$+ c_6k^6z_1^{33}z_2^{11} + c_7k^{15}z_1^{11}z_2^{33} = 0$$

and

$$c_1 + c_3k = 0.$$

The value of k from this equation, substituted into equations (8), locates the points on a_1, corresponding to the point on C_2, in the first order neighborhood of 0_{13332}. The coordinates of this point on a_1 are

$$(0,\ c_3,\ 0,\ 0,\ 0,\ 0,\ 0,\ -c_1).$$

The Σ_2 curve on F in S_7 passes thru the point $O_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and O_1'. This tangent line is represented by

$$X_3 = X_4 = X_5 = X_6 = X_7 = 0,$$

$$c_1X_2 + c_3X_8 = 0.$$

Hence O_1' is a simple point for the Σ_2 curve. It passes once thru O_1' along a line in the tangent plane.

Images of Curves $|C_3|$ on O_1'

Consider the curves $|C_3|$, whose equation is

$$d_1x_1^8x_2^3 + d_2x_1^6x_3^5 + d_3x_1^5x_2^4x_3^2 + d_4x_1^3x_2x_3^7 + d_5x_1^2x_2^5x_3^4 + d_6x_1x_2^9x_3$$
$$+ d_7x_2^2x_3^9 = 0.$$

Applying the transformation V, then U, and finally V again gives

[A] $$d_1z_1^{13}z_2^3 + d_2z_1^{14}z_3^2 + d_3z_1^9z_2^4z_3^3 + d_4z_1^{10}z_2z_3^5 + d_5z_1^5z_2^5z_3^6 + d_6z_2^9z_3^7$$
$$+ d_7z_1^8z_2^2z_3^8 = 0.$$

[B] $$d_1z_1^{20}z_2 + d_2z_1^{19}z_3^2 + d_3z_1^{13}z_2^5z_3^3 + d_4z_1^{12}z_2^4z_3^5 + d_5z_1^6z_2^9z_3^6$$
$$+ d_6z_2^{14}z_3^7 + d_7z_1^5z_2^8z_3^8 = 0.$$

(11) $$z_1^{33}(d_1z_2+d_2z_3) + d_3z_1^{22}z_2^5z_3^7 + d_4z_1^{22}z_2^4z_3^8 + d_5z_1^{11}z_2^9z_3^{14} + d_6z_2^{14}z_3^{20}$$
$$+ d_7z_1^{11}z_2^8z_3^{15} = 0.$$

This shows that the curve C_3 passes doubly thru O_{13}, simply thru O_{132}, and simply thru O_{1323} in the first, second, and third order neighborhoods of O_1.

To approach the point $(z_2 = z_3 = 0)$ along the curve (11), substitute $z_3 = kz_2$ into its equation and allow z_2 to approach zero. The results are

$$z_1^{33}(d_1 z_2 + d_2 k z_2) + d_3 k^7 z_1^{22} z_2^{12} + d_4 k^8 z_1^{22} z_2^{12} + d_5 k^{14} z_1^{11} z_2^{23}$$
$$+ d_6 k^{20} z_2^{34} + d_7 k^{15} z_1^{11} z_2^{23} = 0.$$

$$z_1^{33}(d_1 + d_2 k) + d_3 k^7 z_1^{22} z_2^{11} + d_4 k^8 z_1^{22} z_2^{11} + d_5 k^{14} z_1^{11} z_2^{22}$$
$$+ d_6 k^{20} z_2^{33} + d_7 k^{15} z_1^{11} z_2^{22} = 0,$$

and

$$d_1 + d_2 k = 0.$$

The value of k from this equation, substituted into equation (8), locates the point on a_1, corresponding to the point on C_3, in the first order neighborhood of O_{1323}. The coordinates of this point on a_1 are

$$(0, d_2, 0, 0, 0, 0, -d_1).$$

The Σ_3 curve on F in S_7 passes thru the point $O_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and O_1'. This tangent line is represented by

$$X_3 = X_4 = X_5 = X_6 = X_7 = 0$$

$$d_1 X_2 + d_2 X_8 = 0.$$

Hence O_1' is a simple point for the Σ_3 curve. It passes one time thru O_1' along a line in the tangent plane.

Images of Curves $|C_4|$ at O_1'

Consider the system of curves $|C_4|$,

$$e_1x_1^8x_3^3 + e_2x_1^7x_2^4 + e_3x_1^5x_2x_3^5 + e_4x_1^4x_2^5x_3^2 + e_5x_1^2x_2^2x_3^7 + e_6x_1^6x_2^4x_3 + e_7x_2^{10}x_3 = 0.$$

Applying the transformation U once and V twice gives

[A] $$e_1z_1^{10}z_3^3 + e_2z_1^{12}z_2 + e_3z_1^5z_2^3z_3^5 + e_4z_1^7z_2^4z_3^2 + e_5z_2^6z_3^7 + e_6z_1^2z_2^7z_3^4 + e_7z_1^4z_2^8z_3 = 0.$$

[B] $$e_1z_1^{16}z_3^2 + e_2z_1^{17}z_2 + e_3z_1^8z_2^3z_3^7 + e_4z_1^9z_2^4z_3^5 + e_5z_2^6z_3^{12} + e_6z_1^7z_2^{10}z_3 + e_7z_1^2z_2^8z_3^8 = 0.$$

(12) $$z_1^{22}(e_1z_3+e_2z_2) + e_3z_1^{11}z_2^3z_3^9 + e_4z_1^{11}z_2^4z_3^8 + e_5z_2^6z_3^{17} + e_6z_2^7z_3^{16} + e_7z_2^8z_3^{15} = 0.$$

Thus the curve (12) has a simple point at the point $(z_2 = z_3 = 0)$, and the tangent to this curve at this point has for its equation

$$e_1z_3 + e_2z_2 = 0.$$

The points O_{12}, O_{123}, and O_{1233} are simple. Thus there is one simple variable point in the first order neighborhood of O_{1233}.

To approach the point $(z_2 = z_3 = 0)$ along the curve (12), substitute $z_3 = kz_2$ into its equation and allow z_2 to approach zero. The results are

$$z_1^{22}(e_1kz_2 + e_2z_2) + e_3k^9 z_1^{11} z_2^{12} + e_4k^8 z_1^{11} z_2^{12} + e_5k^{17} z_2^{23}$$
$$+ e_6k^{16} z_2^{23} + e_7k^{15} z_2^{23} = 0.$$

$$z_1^{22}(e_1k + e_2) + e_3k^9 z_1^{11} z_2^{11} + e_4k^8 z_1^{11} z_2^{11} + e_5k^{11} z_2^{22} + e_6k^{16} z_2^{22}$$
$$+ e_7k^{15} z_2^{22} = 0,$$

and

$$e_1k + e_2 = 0.$$

The value of k from this equation, substituted into equation (6), locates the point on G_1, corresponding to the point on C_4, in the first order neighborhood of O_{1233}. The coordinates of this point on G_1 are

$$(0,\ e_2^2,\ 0,\ -e_1e_2,\ 0,\ 0,\ e_1^2,\ 0).$$

The curve Σ_4 on F in S_7 passes thru the point $O_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and O_1'. This tangent line is represented by

$$X_3 = X_5 = X_6 = X_8 = 0,$$

$$e_1X_2 + e_2X_4 = 0,$$

$$e_1X_4 + e_2X_7 = 0.$$

Hence O_1' is a simple point for the Σ_4 curves. They pass thru O_1' along an element of the quadric tangent cone.

Images of Curves $|C_5|$ at O_1'

Consider the system of curves $|C_5|$,

$$f_1x_1^{10}x_3 + f_2x_1^7x_2x_3^3 + f_3x_1^6x_2^5 + f_4x_1^4x_2^2x_3^5 + f_5x_1^3x_2^6x_3^2 + f_6x_1x_2^3x_3^7$$

$$+ f_7x_2^7x_3^4 = 0.$$

Applying the transformation U four times gives

[A] $$f_1z_1^{15}z_3 + f_2z_1^{10}z_2^3z_3^3 + f_3z_1^{12}z_2^4 + f_4z_1^5z_2^6z_3^5 + f_5z_1^7z_2^7z_3^2 + f_6z_2^9z_3^7$$

$$+ f_7z_1^2z_2^{10}z_3^4 = 0.$$

[B] $$f_1z_1^{21}z_3 + f_2z_1^{14}z_2^5z_3^3 + f_3z_1^{19}z_2^3 + f_4z_1^7z_2^{10}z_3^5 + f_5z_1^{12}z_2^8z_3^2$$

$$+ f_6z_2^{15}z_3^7 + f_7z_1^5z_2^{13}z_3^4 = 0.$$

[C] $$f_1z_1^{27}z_3 + f_2z_1^{18}z_2^7z_3^3 + f_3z_1^{26}z_2^2 + f_4z_1^9z_2^{14}z_3^5 + f_5z_1^{17}z_2^9z_3^2 +$$

$$+ f_6z_2^{21}z_3^7 + f_7z_1^8z_2^{16}z_3^4 = 0.$$

(13) $$z_1^{33}(f_1z_3+f_3z_2) + f_2z_1^{22}z_2^9z_3^3 + f_4z_1^{11}z_2^{18}z_3^5 + f_5z_1^{22}z_2^{10}z_3^2 + f_6z_2^{27}z_3^7$$

$$+ f_7z_1^{11}z_2^{19}z_3^4 = 0.$$

Thus the curve (13) has a simple point at the point $(z_2 = z_3 = 0)$, and the tangent to this curve at this point has for its equation

$$f_1 z_3 + f_3 z_2 = 0.$$

The points O_{12}, O_{122}, O_{1222}, and O_{12222} are simple. Thus there is one simple variable point in the first order neighborhood of O_{12222}.

To approach the point $(z_2 = z_3 = 0)$ along the curve (13), substitute $z_3 = kz_2$ into its equation and allow z_2 to approach zero. The results are

$$z_1^{33}(f_1 k z_2 + f_3 z_2) + f_2 k^3 z_1^{22} z_2^{12} + f_4 k^5 z_1^{11} z_2^{23} + f_5 k^2 z_1^{22} z_2^{12}$$
$$+ f_6 k^7 z_2^{34} + f_7 k^4 z_1^{11} z_2^{23} = 0.$$

$$z_1^{33}(f_1 k + f_3) + f_2 k^3 z_1^{22} z_2^{11} + f_4 k^5 z_1^{11} z_2^{22} + f_5 k^2 z_1^{22} z_2^{11} + f_6 k^7 z_2^{33}$$
$$+ f_7 k^4 z_1^{11} z_2^{22} = 0,$$

and

$$f_1 k + f_3 = 0.$$

The value of k from this equation, substituted into equation (6), locates the point on G_1, corresponding to the point on $|C_5|$, in the first order neighborhood of O_{12222}. The coordinates of this point on G_1 are

$$(0,\ f_3^2,\ 0,\ -f_1 f_3,\ 0,\ 0,\ f_1^2,\ 0).$$

The curve Σ_5 on F in S_7 passes thru the point $O_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and O_1'. This tangent line is represented by

$$X_3 = X_5 = X_6 = X_8 = 0,$$

$$f_1X_2 + f_3X_4 = 0,$$

$$f_3X_7 + f_1X_4 = 0.$$

Hence O_1' is a simple point for the Σ_5 curve. It passes thru O_1' along an element of the quadric tangent cone.

Images of Curves $|C_6|$ at O_1'

Consider the system of curves $|C_6|$,

$$g_1X_1^9X_2X_3^6 + g_2X_1^6X_2^2X_3^3 + g_3X_1^5X_2^6 + g_4X_1^3X_2^3X_3^5 + g_5X_1^2X_2^7X_3^2 + g_6X_1^1X_3^{10}$$
$$+ g_7X_2^4X_3^7 = 0.$$

Applying the transformation U four times gives

[A] $$g_1z_1^{17}z_3 + g_2z_1^{12}z_2^3z_3^3 + g_3z_1^{14}z_2^4 + g_4z_1^7z_2^6z_3^5 + g_5z_1^9z_2^7z_3^2 + g_6z_2^8z_3^{10}$$
$$+ g_7z_1^2z_2^9z_3^7 = 0.$$

[B] $$g_1z_1^{26}z_3 + g_2z_1^{19}z_2^5z_3^3 + g_3z_1^{24}z_2^3 + g_4z_1^{12}z_2^{10}z_3^5 + g_5z_1^{17}z_2^8z_3^2$$
$$+ g_6z_2^{17}z_3^{10} + g_7z_1^5z_2^{15}z_3^7 = 0.$$

[C] $$g_1 z_1^{27} z_3 + g_2 z_1^{18} z_2^7 z_3^3 + g_3 z_1^{26} z_2^2 + g_4 z_1^9 z_2^{14} z_3^5 + g_5 z_1^{17} z_2^9 z_3^2$$
$$+ g_6 z_1^9 z_2^9 z_3^{10} + g_7 z_2^{21} z_3^7 = 0.$$

(14) $$z_1^{33}(g_1 z_3 + g_3 z_2) + g_2 z_1^{22} z_2^9 z_3^3 + g_4 z_1^{11} z_2^{18} z_3^5 + g_5 z_1^{22} z_2^{10} z_3^2$$
$$+ g_6 z_1^6 z_2^{18} z_3^{10} + g_7 z_2^{27} z_3^7 = 0.$$

Thus the curve (14) has a simple point at the point ($z_2 = z_3 = 0$), and the tangent to this curve at this point has for its equation

$$g_1 z_3 + g_3 z_2 = 0.$$

The points O_{12}, O_{122}, O_{1222}, and O_{12222} are simple, and there is one simple variable point in the first order neighborhood of O_{12222}.

To approach the point ($z_2 = z_3 = 0$) along the curve (14), substitute $z_3 = kz_2$ into its equation and allow z_2 to approach zero. The results are

$$z_1^{33}(g_1 k z_2 + g_3 z_2) + g_2 k^3 z_1^{22} z_2^{12} + g_4 k^5 z_1^{11} z_2^{23} + g_5 k^2 z_1^{22} z_2^{12}$$
$$+ g_6 k^{10} z_1^6 z_2^{18} + g_7 k^7 z_2^{34} = 0.$$

$$z_1^{33}(g_1 k + g_3) + g_2 k^3 z_1^{22} z_2^{11} + g_4 k^5 z_1^{11} z_2^{22} + g_5 k^2 z_1^{22} z_2^{11}$$
$$+ g_6 k^{10} z_1^6 z_2^{17} + g_7 k^7 z_2^{33} = 0,$$

and

$$g_1 k + g_3 = 0.$$

The value of k from this equation, substituted into equation (6), locates the point on G_1, corresponding to the point on $|C_6|$, in the first order neighborhood of O_{12222}. The coordinates of this point on G_1 are

$$(0,\ g_3^2,\ 0,\ -g_1g_3,\ 0,\ 0,\ g_1^2,\ 0).$$

The curve Σ_6 on F in S_7 passes thru the point $O_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and O_1'. This tangent line is represented by

$$X_3 = X_5 = X_6 = X_8 = 0,$$

$$g_1X_2 + g_3X_4 = 0,$$

$$g_3X_7 + g_1X_4 = 0.$$

Applying the transformation V eight times gives

[A] $$g_1z_1^{13}z_2 + g_2z_1^9z_2^2z_3^3 + g_3z_1^4z_2^6z_3^4 + g_4z_1^5z_2^3z_3^6 + g_5z_2^7z_3^7 + g_6z_1^6z_3^8 + g_7z_1z_2^4z_3^9 = 0.$$

[B] $$g_1z_1^{19}z_2 + g_2z_1^{14}z_2^2z_3^4 + g_3z_1^5z_2^6z_3^9 + g_4z_1^9z_2^3z_3^8 + g_5z_2^7z_3^{15} + g_6z_1^{13}z_3^7 + g_7z_1^4z_2^4z_3^{12} = 0.$$

[C] $$g_1z_1^{25}z_2 + g_2z_1^{19}z_2^2z_3^5 + g_3z_1^6z_2^8z_3^{14} + g_4z_1^{13}z_2^3z_3^{10} + g_5z_2^7z_3^{19} + g_6z_1^{20}z_3^6 + g_7z_1^7z_2^4z_3^{15} = 0.$$

[D] $$g_1z_1^{31}z_2 + g_2z_1^{24}z_2^2z_3^6 + g_3z_1^7z_2^8z_3^{19} + g_4z_1^{17}z_2^3z_3^{12} + g_5z_2^7z_3^{25} + g_6z_1^{27}z_3^5 + g_7z_1^{10}z_2^4z_3^{18} = 0.$$

[E] $$g_1 z_1^{37} z_2 + g_2 z_1^{29} z_2^2 z_3^7 + g_3 z_1^8 z_2^6 z_3^{24} + g_4 z_1^{21} z_2^3 z_3^{14} + g_5 z_2^7 z_3^{31}$$
$$+ g_6 z_1^{34} z_3^4 + g_7 z_1^{15} z_2^4 z_3^{21} = 0.$$

[F] $$g_1 z_1^{43} z_2 + g_2 z_1^{34} z_2^2 z_3^8 + g_3 z_1^9 z_2^6 z_3^{29} + g_4 z_1^{25} z_2^3 z_3^{16} + g_5 z_2^7 z_3^{37}$$
$$+ g_6 z_1^{41} z_3^3 + g_7 z_1^{16} z_2^4 z_3^{24} = 0.$$

[G] $$g_1 z_1^{49} z_2 + g_2 z_1^{39} z_2^2 z_3^9 + g_3 z_1^{10} z_2^6 z_3^{34} + g_4 z_1^{29} z_2^3 z_3^{18} + g_5 z_2^7 z_3^{43}$$
$$+ g_6 z_1^{48} z_3^2 + g_7 z_1^{19} z_2^4 z_3^{27} = 0.$$

(15) $$z_1^{55}(g_1 z_2 + g_6 z_3) + g_2 z_1^{44} z_2^2 z_3^{10} + g_3 z_1^{11} z_2^6 z_3^{39} + g_4 z_1^{33} z_2^3 z_3^{20}$$
$$+ g_5 z_2^7 z_3^{49} + g_7 z_1^{22} z_2^4 z_3^{30} = 0.$$

Thus the curve (15) has a simple point at the point $(z_2 = z_3 = 0)$, and the tangent to this curve at this point has for its equation

$$g_1 z_2 + g_6 z_3 = 0.$$

The point $O_{15(8)}$ is therefore simple and there is one simple variable point in the first order neighborhood of $O_{15(8)}$. We note here that the points O_{15}, O_{155}, O_{1555}, $O_{15(4)}$, $O_{15(5)}$, $O_{15(6)}$, and $O_{15(7)}$ are simple, also.

To approach the point $(z_2 = z_3 = 0)$ along the curve (15), substitute $z_3 = kz_2$ into its equation and allow z_2 to approach zero. The results are

$$z_1^{55}(g_1z_2+g_6kz_2) + g_2k^{10}z_1^{44}z_2^{12} + g_3k^{39}z_1^{11}z_2^{45} + g_4k^{20}z_1^{33}z_2^{23}$$

$$+ g_5k^{49}z_2^{56} + g_7k^{30}z_1^{22}z_2^{34} = 0.$$

$$z_1^{55}(g_1+g_6k) + g_2k^{10}z_1^{44}z_2^{11} + g_3k^{39}z_1^{11}z_2^{44} + g_4k^{20}z_1^{33}z_2^{22}$$

$$+ g_5k^{49}z_2^{55} + g_7k^{30}z_1^{22}z_2^{33} = 0,$$

and

$$g_1 + g_6k = 0.$$

The value of k from this equation, substituted into equation (8), locates the point on a_1 corresponding to the point on C_6, in the first order neighborhood of $O_{15(8)}$. The coordinates of this point on a_1 are

$$(0, g_6, 0, 0, 0, 0, 0, -g_1).$$

Here the curve Σ_6 on F in S_7 passes thru the point $O_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and O_1'. This tangent line is represented by

$$X_3 = X_4 = X_5 = X_8 = X_7 = 0$$

and

$$g_1X_2 + g_6X_8 = 0.$$

Hence, O_1' is a double point for Σ_6 curves. One branch goes through O_1' along an element of the quadric tangent cone, while the other branch goes through along a line in the tangent plane. This double point is a node, since the two lines are distinct.

Images of Curves $|C_7|$ at O_1'

Consider the system of curves $|C_7|$,

$$h_1x_1^8x_2^2x_3 + h_2x_1^5x_2^3x_3^3 + h_3x_1^4x_2^7 + h_4x_1^3x_3^8 + h_5x_1^2x_2^4x_3^5 + h_6x_1^8x_2^2x_3^2$$
$$+ h_7x_2^8x_3^{10} = 0.$$

Applying the transformation V three times, followed by U once gives

[A] $$h_1z_1^{13}z_2^2 + h_2z_1^9z_2^3z_3^3 + h_3z_1^4z_2^7z_3^4 + h_4z_1^{10}z_3^5 + h_5z_1^5z_2^4z_3^6 + h_6z_2^8z_3^7$$
$$+ h_7z_1^6z_2z_3^8 = 0.$$

[B] $$h_1z_1^{19}z_2^2 + h_2z_1^{14}z_2^3z_3^4 + h_3z_1^5z_2^7z_3^9 + h_4z_1^{18}z_3^3 + h_5z_1^9z_2^4z_3^8 + h_6z_2^8z_3^{13}$$
$$+ h_7z_1^{13}z_2z_3^7 = 0.$$

[C] $$h_1z_1^{25}z_2^2 + h_2z_1^{19}z_2^3z_3^5 + h_3z_1^6z_2^7z_3^{14} + h_4z_1^{26}z_3 + h_5z_1^{13}z_2^4z_3^{10}$$
$$+ h_6z_2^8z_3^{19} + h_7z_1^{20}z_2z_3^6 = 0.$$

(16) $$z_1^{44}(h_1z_2+h_4z_3) + h_2z_1^{33}z_2^7z_3^5 + h_3z_1^{11}z_2^{20}z_3^{14} + h_5z_1^{22}z_2^{15}z_3^{10}$$
$$+ h_6z_2^{26}z_3^{19} + h_7z_1^{33}z_2^6z_3^6 = 0.$$

Thus the curve (16) has a simple point at the point $(z_2 = z_3 = 0)$, and the tangent to this curve at this point has for its equation

$$h_1z_2 + h_4z_3 = 0.$$

The points 0_{13}, 0_{133}, and 0_{1333} are double while the point 0_{13332} is simple, and there is one simple variable point in the first order neighborhood of 0_{13332}.

To approach the point $(z_2 = z_3 = 0)$ along the curve (16), substitute $z_3 = kz_2$ into its equations and allow z_2 to approach zero. The results are

$$z_1^{44}(h_1 z_2 + h_4 k z_2) + h_2 k^5 z_1^{33} z_2^{12} + h_3 k^{14} z_1^{11} z_2^{34} + h_5 k^{10} z_1^{22} z_2^{23}$$
$$+ h_6 k^{19} z_2^{45} + h_7 k^6 z_1^{33} z_2^{12} = 0.$$

$$z_1^{44}(h_1 + h_4 k) + h_2 k^5 z_1^{33} z_2^{11} + h_3 k^{14} z_1^{11} z_2^{33} + h_5 k^{10} z_1^{22} z_2^{22}$$
$$+ h_6 k^{19} z_2^{44} + h_7 k^6 z_1^{33} z_2^{12} = 0,$$

and

$$h_1 + h_4 k = 0.$$

The value of k from this equation, substituted into equation (8), locates the point on a_1, corresponding to the point on C_7, in the first order neighborhood of 0_{13332}. The coordinates of this point on a_1 are

$$(0, h_4, 0, 0, 0, 0, 0, -h_1).$$

The curve Σ_7 on F in S_7 passes thru the point $0_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and $0_1'$. This tangent line is represented by

$$X_3 = X_4 = X_5 = X_6 = X_7 = 0$$

$$h_1 X_2 + h_4 X_8 = 0.$$

Now, applying the transformation U four times to $|C_7|$ gives

[A] $$h_1 z_1^{17} z_3 + h_2 z_1^{12} z_2^3 z_3^3 + h_3 z_1^{14} z_2^4 + h_4 z_1^5 z_2^5 z_3^8 + h_5 z_1^7 z_2^6 z_3^5 + h_6 z_1^9 z_2^7 z_3^2 + h_7 z_2^8 z_3^{10} = 0.$$

[B] $$h_1 z_1^{26} z_3 + h_2 z_1^{19} z_2^5 z_3^3 + h_3 z_1^{24} z_2^3 + h_4 z_1^7 z_2^{12} z_3^8 + h_5 z_1^{12} z_2^{10} z_3^5 + h_6 z_1^{17} z_2^8 z_3^2 + h_7 z_2^{17} z_3^{10} = 0.$$

[C] $$h_1 z_1^{35} z_3 + h_2 z_1^{26} z_2^7 z_3^3 + h_3 z_1^{34} z_2^2 + h_4 z_1^9 z_2^{19} z_3^8 + h_5 z_1^{17} z_2^{14} z_3^5 + h_6 z_1^{25} z_2^9 z_3^2 + h_7 z_2^{26} z_3^{10} = 0.$$

(17) $$z_1^{44}(h_1 z_3 + h_3 z_2) + h_2 z_1^{33} z_2^9 z_3^3 + h_4 z_1^{11} z_2^{26} z_3^8 + h_5 z_1^{22} z_2^{18} z_3^5 + h_6 z_1^{33} z_2^{10} z_3^2 + h_7 z_2^{35} z_3^{10} = 0.$$

Thus the curve (17) has a simple point at the point $(z_2 = z_3 = 0)$ and the tangent to this curve at this point has for its equation

$$h_1 z_3 + h_3 z_2 = 0.$$

The points O_{12}, O_{122}, O_{1222}, and O_{12222} are simple, and there is one simple variable point in the first order neighborhood of O_{12222}.

To approach the point $(z_2 = z_3 = 0)$ along the curve (17), substitute $z_3 = kz_2$ into its equation and allow z_2 to approach zero.

$$z_1^{44}(h_1kz_2+h_3z_2) + h_2k^3z_1^{33}z_2^{12} + h_4k^8z_1^{11}z_2^{34} + h_5k^5z_1^{22}z_2^{23}$$

$$+ h_6k^2z_1^{33}z_2^{12} + h_7k^{10}z_2^{45} = 0.$$

$$z_1^{44}(h_1k+h_3) + h_2k^3z_1^{33}z_2^{11} + h_4k^8z_1^{11}z_2^{33} + h_5k^5z_1^{22}z_2^{22} + h_6k^2z_1^{33}z_2^{11}$$

$$+ h_7k^{10}z_2^{44} = 0$$

and

$$h_1k + h_3 = 0.$$

The value of k from this equation, substituted into equation (6), locates the point on G_1, corresponding to the point on $|C_7|$, in the first order neighborhood of O_{12222}. The coordinates of this point on G_1 are

$$(0,\ h_3^2,\ 0,\ -h_1h_3,\ 0,\ 0,\ h_1^2,\ 0).$$

The curve Σ_7 on F in S_7 passes thru the point $O_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and O_1'. This tangent line is represented by

$$X_3 = X_5 = X_6 = X_8 = 0,$$

$$h_1X_2 + h_3X_4 = 0,$$

$$h_1X_4 + h_3X_7 = 0.$$

Hence O_1' is a double point for the curve Σ_7. One branch goes thru O_1' along an element of the quadric tangent cone, while the other branch goes thru once along a line in the tangent plane.

Images of Curves $|C_8|$ at O_1'

Consider the system of curves $|C_8|$,

$$j_1x_1^7x_2^3x_3 + j_2x_1^5x_3^6 + j_3x_1^4x_2^4x_3^3 + j_4x_1^3x_2^8 + j_5x_1^2x_2x_3^8 + j_6x_1x_2^5x_3^5$$
$$+ j_7x_2^9x_3^2 = 0.$$

Applying the transformation U four times gives

[A] $$j_1z_1^{12}z_3 + j_2z_1^5z_2^2z_3^6 + j_3z_1^7z_2^3z_3^3 + j_4z_1^9z_2^4 + j_5z_2^5z_3^8 + j_6z_1^2z_2^6z_3^5$$
$$+ j_7z_1^4z_2^7z_3^2 = 0.$$

[B] $$j_1z_1^{19}z_3 + j_2z_1^7z_2^7z_3^6 + j_3z_1^{12}z_2^5z_3^3 + j_4z_1^{17}z_2^3 + j_5z_2^{12}z_3^8$$
$$+ j_6z_1^5z_2^{10}z_3^5 + j_7z_1^{10}z_2^8z_3^2 = 0.$$

[C] $$j_1z_1^{26}z_3 + j_2z_1^9z_2^{12}z_3^6 + j_3z_1^{17}z_2^7z_3^3 + j_4z_1^{25}z_2^2 + j_5z_2^{19}z_3^8$$
$$+ j_6z_1^8z_2^{14}z_3^5 + j_7z_1^{16}z_2^9z_3^2 = 0.$$

(18) $$z_1^{33}(j_1z_3+j_4z_2) + j_2z_1^{11}z_2^{17}z_3^6 + j_3z_1^{22}z_2^9z_3^3 + j_5z_2^{26}z_3^8$$
$$+ j_6z_1^{11}z_2^{18}z_3^5 + j_7z_1^{22}z_2^{10}z_3^2 = 0.$$

Thus the curve (18) has a simple point at the point $(z_2 = z_3 = 0)$, and the tangent to this curve at this point has for its equation $j_1c_3 + j_4c_2 = 0$.

The points O_{12}, O_{122}, O_{1222}, and O_{12222} are simple, and there is one simple variable point in the first order neighborhood of O_{12222}.

To approach the point ($z_2 = z_3 = 0$) along the curve (18), substitute $z_3 = kz_2$ into its equation and allow z_2 to approach zero. The results are

$$z_1^{33}(j_1kz_2+j_4z_2) + j_2k^6z_1^{11}z_2^{23} + j_3k^3z_1^{22}z_2^{12} + j_5k^8z_2^{34} + j_6k^5z_1^{11}z_2^{23}$$

$$+ j_7k^2z_1^{22}z_2^{12} = 0.$$

$$z_1^{33}(j_1k+j_4) + j_2k^6z_1^{11}z_2^{22} + j_3k^3z_1^{22}z_2^{11} + j_5k^8z_2^{33} + j_6k^5z_1^{11}z_2^{22}$$

$$+ j_7k^2z_1^{22}z_2^{11} = 0$$

and

$$j_1k + j_4 = 0.$$

The value of k from this equation, substituted into equation (6), locates the point on G_1, corresponding to the point on C_8, in the first order neighborhood of O_{12222}. The coordinates of this point on G_1 are

$$(0,\ j_4^2,\ 0,\ -j_1j_4,\ 0,\ 0,\ j_1^2,\ 0).$$

The curve Σ_8 on F in S_7 passes thru the point $O_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and O_1'. The tangent line is represented by

$$X_3 = X_5 = X_6 = X_8 = 0,$$

$$j_1X_2 + j_4X_4 = 0,$$

$$j_1X_4 + j_4X_7 = 0.$$

Applying the transformation V, then U, then V gives

$$[A] \qquad j_1 z_1^{13} z_2^{3} + j_2 z_1^{14} z_3^{2} + j_3 z_1^{9} z_2^{4} z_3^{3} + j_4 z_1^{4} z_2^{8} z_3^{4} + j_5 z_1^{10} z_2^{2} z_3^{5} + j_6 z_1^{5} z_2^{5} z_3^{6}$$

$$+ j_7 z_2^{9} z_3^{7} = 0.$$

$$[B] \qquad j_1 z_1^{20} z_2 + j_2 z_1^{19} z_3^{2} + j_3 z_1^{13} z_2^{5} z_3^{3} + j_4 z_1^{7} z_2^{10} z_3^{4} + j_5 z_1^{12} z_2^{4} z_3^{5}$$

$$+ j_6 z_1^{6} z_2^{9} z_3^{6} + j_7 z_2^{14} z_3^{7} = 0.$$

$$(19) \qquad z_1^{33}(j_1 z_2 + j_2 z_3) + j_3 z_1^{22} z_2^{5} z_3^{7} + j_4 z_1^{11} z_2^{10} z_3^{13} + j_5 z_1^{22} z_2^{4} z_3^{8} + j_6 z_1^{11} z_2^{9} z_3^{14}$$

$$+ j_7 z_2^{14} z_3^{20} = 0.$$

Thus the curve (19) has a simple point at the point $(z_2 = z_3 = 0)$, and the tangent to this curve at this point has for its equation

$$j_1 z_2 + j_2 z_3 = 0.$$

The point O_{13} is double while the point O_{132} and the point O_{1323} are simple, and there is one simple variable point in the first order neighborhood of O_{1323}.

To approach the point $(z_2 = z_3 = 0)$ along the curve (19), substitute $z_3 = kz_2$ into its equation and allow z_2 to approach zero. The results are

$$z_1^{33}(j_1 z_2 + j_2 k z_2) + j_3 k^{7} z_1^{22} z_2^{12} + j_4 k^{15} z_1^{11} z_2^{23} + j_5 k^{8} z_1^{22} z_2^{12}$$

$$+ j_6 k^{14} z_1^{11} z_2^{23} + j_7 k^{20} z_2^{34} = 0.$$

$$z_1^{33}(j_1+j_2k) + j_3k^7z_1^{22}z_2^{11} + j_4k^{13}z_1^{11}z_2^{22} + j_5k^8z_1^{22}z_2^{11}$$

$$+ j_6k^{14}z_1^{11}z_2^{22} + j_7k^{20}z_2^{33} = 0,$$

and

$$j_1 + j_2k = 0.$$

The value of k from this equation, substituted into equation (8), locates the point on a_1, corresponding to the point on C_8, in the first order neighborhood of O_{1323}. The coordinates of this point on a_1 are

$$(0,\ j_2,\ 0,\ 0,\ 0,\ 0,\ -j_1).$$

The curve Σ_8 on F in S_7 passes thru the point $O_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and O_1'. This tangent line is represented by

$$X_3 = X_4 = X_5 = X_6 = X_7 = 0$$
$$j_1X_2 + j_2X_8 = 0.$$

Hence O_1' is a double point for the Σ_8 curve. One branch goes thru O_1' along an element of the quadric tangent cone, while the other branch goes thru one time along a line in the tangent plane.

Images of Curves $|C_9|$ at O_1'

Consider the system of curves $|C_9|$,

$$k_1x_1^7x_3^4 + k_2x_1^6x_2^4x_3 + k_3x_1^4x_2x_3^6 + k_4x_1^3x_2^5x_3^3 + k_5x_1^2x_2^9 + k_6x_1x_2^2x_3^8$$

$$+ k_7x_2^6x_3^5 = 0.$$

Applying the transformation U four times gives

[A] $$k_1 z_1^{10} z_3^4 + k_2 z_1^{12} z_2 z_3 + k_3 z_1^5 z_2^3 z_3^6 + k_4 z_1^7 z_2^4 z_3^3 + k_5 z_1^9 z_2^5 + k_6 z_2^6 z_3^8$$
$$+ k_7 z_1^2 z_2^7 z_3^5 = 0.$$

[B] $$k_1 z_1^{14} z_2^2 z_3^4 + k_2 z_1^{19} z_3 + k_3 z_1^7 z_2^7 z_3^6 + k_4 z_1^{12} z_2^5 z_3^3 + k_5 z_1^{17} z_2^3$$
$$+ k_6 z_1^{12} z_2^8 + k_7 z_1^5 z_2^{10} z_3^5 = 0.$$

[C] $$k_1 z_1^{18} z_2^5 z_3^4 + k_2 z_1^{26} z_3 + k_3 z_1^9 z_2^{12} z_3^6 + k_4 z_1^{17} z_2^7 z_3^3 + k_5 z_1^{25} z_2^2$$
$$+ k_6 z_2^{19} z_3^8 + k_7 z_1^8 z_2^{14} z_3^5 = 0.$$

(20) $$z_1^{33}(k_2 z_3 + k_5 z_2) + k_1 z_1^{22} z_2^8 z_3^4 + k_3 z_1^{11} z_2^{17} z_3^6 + k_4 z_1^{22} z_2^9 z_3^3$$
$$+ k_6 z_2^{26} z_3^8 + k_7 z_1^{11} z_2^{18} z_3^5 = 0.$$

Thus the curve (20) has a simple point at the point $(z_2 = z_3 = 0)$ and the tangent to this curve at this point has for its equation

$$k_2 z_3 + k_5 z_2 = 0.$$

The point O_{12} is double, while the points O_{122}, O_{1222}, and O_{12222} are simple. There is one simple variable point in the first order neighborhood of O_{12222}.

To approach the point $(z_2 = z_3 = 0)$ along the curve (20), substitute $z_3 = kz_2$ into its equation and allow z_2 to approach zero. The results are

$$z_1^{33}(k_2kz_2+k_5z_2) + k_1k^4z_1^{22}z_2^{12} + k_3k^6z_1^{11}z_2^{23} + k_4k^3z_1^{22}z_2^{12}$$

$$+ k_6k^8z_2^{34} + k_7k^5z_1^{11}z_2^{23} = 0.$$

$$z_1^{33}(k_2k+k_5) + k_1k^4z_1^{22}z_2^{11} + k_3k^6z_1^{11}z_2^{22} + k_4k^3z_1^{22}z_2^{11} + k_6k^8z_2^{33}$$

$$+ k_7k^5z_1^{11}z_2^{22} = 0.$$

and

$$k_2k + k_5 = 0.$$

The value of k from this equation, substituted into equation (6), locates the point on G_1, corresponding to the point on C_9, in the first order neighborhood of O_{12222}. The coordinates of this point on G_1 are

$$(0,\ k_5^2,\ 0,\ -k_2k_5,\ 0,\ 0,\ k_2^2,\ 0).$$

The curve Σ_9 on F in S_7 passes thru the point $O_1'(1,0,0,0,0,0,0,0)$ along the tangent line determined by the above point and O_1'. This tangent line is represented by

$$X_3 = X_5 = X_6 = X_8 = 0,$$

$$k_2X_2 + k_5X_4 = 0,$$

$$k_2X_4 + k_5X_7 = 0.$$

Hence O_1' is a simple point for the Σ_9 curve. It passes one time thru O_1' along an element of the quadric tangent cone.

Images of Curves $|C_{10}|$ at O_1'

Consider the system of curves $|C_{10}|$,

$$l_1x_1^9x_3^2 + l_2x_1^6x_2x_3^4 + l_3x_1^5x_2^5x_3 + l_4x_1^3x_2^2x_3^6 + l_5x_1^2x_2^6x_3^3 + l_6x_1x_2^{10}$$

$$+ l_7x_2^3x_3^8 = 0.$$

Applying the transformation U four times gives

[A] $$l_1z_1^{15}z_3^2 + l_2z_1^{10}z_2^3z_3^4 + l_3z_1^{12}z_2^4z_3^1 + l_4z_1^5z_2^6z_3^6 + l_5z_1^7z_2^7z_3^3$$

$$+ l_6z_1^9z_2^8 + l_7z_2^9z_3^8 = 0.$$

[B] $$l_1z_1^{21}z_3^2 + l_2z_1^{14}z_2^5z_3^4 + l_3z_1^{19}z_2^3z_3^1 + l_4z_1^7z_2^{10}z_3^6 + l_5z_1^{12}z_2^8z_3^5$$

$$+ l_6z_1^{17}z_2^6 + l_7z_2^{15}z_3^8 = 0.$$

[C] $$l_1z_1^{27}z_3^2 + l_2z_1^{18}z_2^7z_3^4 + l_3z_1^{26}z_2^2z_3^1 + l_4z_1^9z_2^{14}z_3^6 + l_5z_1^{17}z_2^9z_3^5$$

$$+ l_6z_1^{25}z_2^4 + l_7z_2^{21}z_3^8 = 0.$$

(21) $$z_1^{33}(l_1z_3^2+l_3z_2z_3+l_6z_2^2) + l_2z_1^{22}z_2^9z_3^4 + l_4z_1^{11}z_2^{18}z_3^6 + l_5z_1^{22}z_2^{10}z_3^3$$

$$+ l_7z_2^{27}z_3^8 = 0.$$

This curve has a double point at the point $(z_2 = z_3 = 0)$, and the tangent to the curve at this point has the equation

$$l_1z_3^2 + l_3z_2z_3 + l_6z_2^2 = 0.$$

Thus the point O_{12222}, in the fourth order neighborhood of O_1, is a double point, and there are two simple variable points in the

first order neighborhood of O_{12222}. Furthermore, we note that the points O_{12}, O_{122}, and O_{1222} are double points.

To approach the point ($z_2 = z_3 = 0$) along the curve (21), substitute $z_3 = kz_2$ and allow z_2 to approach zero. The results are

$$z_1^{33}(l_1k^2z_2^2+l_3kz_2^2+ l_6z_2^2) + l_2k^4z_1^{22}z_2^{13} + l_4k^6z_1^{11}z_2^{24} + l_5k^5z_1^{22}z_2^{13}$$

$$+ l_7k^8z_2^{35} = 0.$$

$$z_1^{33}(l_1k^2+l_3k+l_6) + l_2k^4z_1^{22}z_2^{11} + l_4k^6z_1^{11}z_2^{22} + l_5k^3z_1^{22}z_2^{11}$$

$$+ l_7k^8z_2^{33} = 0,$$

and

$$l_1k^2 + l_3k + l_6 = 0.$$

The coordinates, then, of the two points on G_1, corresponding to the two variable points, infinitely near O_{12222}, are therefore

$$(0,\ k_a^2,\ 0,\ k_a,\ 0,\ 0,\ 1,\ 0)$$

and

$$(0,\ k_b^2,\ 0,\ k_b,\ 0,\ 0,\ 1,\ 0),$$

where k_a and k_b are the roots of

$$l_1k^2 + l_3k + l_6 = 0.$$

Members of the system $|2_{10}|$, the images of the system $|C_{10}|$, on F in S_7, has therefore a double point at O_1', and the two tangent lines to the curve at this point lie on the quadric cone

$$X_2X_7 = X_4^2, \qquad X_3 = X_5 = X_6 = X_8 = 0$$

and have for equations

$$X_3 = X_5 = X_6 = X_7 = 0$$
$$l_1X_2 + l_3X_4 + l_6X_7 = 0$$
$$-X_4 + k_aX_7 = 0$$

and

$$X_3 = X_5 = X_6 = X_7 = 0$$
$$l_1X_2 + l_3X_4 + l_6X_7 = 0$$
$$-X_4 + k_bX_7 = 0$$

where the values of k are the roots of

$$l_1k^2 + l_3k + l_6 = 0.$$

Hence O_1' is a double point for the Σ_{10} curve, and it passes twice thru O_1' along two elements of the quadric tangent cone.

Summary

The branch point O_1' on F, the image of O_1, was studied. O_1' was found to be a triple point on F. The tangent elements to the surface F in S_7 at O_1' were a quadric cone and a plane.

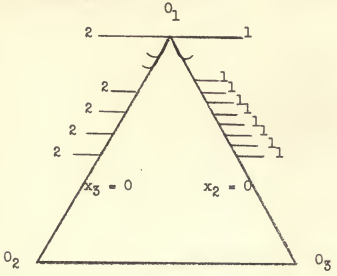
Ten invariant systems of plane algebraic curves of degree eleven were mapped onto the surface F. Their images were ten systems of curves denoted by $|\Sigma_i|$, $(i = 1,2,3, \ldots, 10)$.

$|\Sigma_1|$, $|\Sigma_2|$, $|\Sigma_3|$, $|\Sigma_4|$ and $|\Sigma_5|$ passed simply thru O_1'. $|\Sigma_1|$, $|\Sigma_2|$ and $|\Sigma_3|$ had the tangent in the plane, while $|\Sigma_4|$ and $|\Sigma_5|$ had the tangent line on the quadric cone.

$|\Sigma_6|$, $|\Sigma_7|$ and $|\Sigma_8|$ passed doubly thru O_1', with one tangent line in the plane and a second tangent line on the quadric cone, respectively.

$|\Sigma_9|$ passed simply thru O_1', with a tangent line lying on the quadric cone, while $|\Sigma_{10}|$ passed doubly thru O_1' with two tangent lines lying on the quadric cone.

Here is the triangle of reference.



Explanation of the Reference Triangle

This reference triangle shows the multiplicities of the curves $C_i (i = 1,2,3, \ldots, 10)$ at O_1 and at the points infinitely near O_1, in the invariant directions $x_2 = 0$ and $x_3 = 0$.

The reader will observe from equation (4), near the beginning of this chapter, that the term possessing x_1 to the highest exponent is $a_2 x_1^8 x_2 x_3^2$. Thus O_1 is a triple point with tangent lines $x_2 = 0$, and $x_3 = 0$ counted twice. This information is represented by a "2" to the left of O_1 and a "1" to the right of O_1.

The applications of the quadratic transformations U and V to the curves $C_i (i = 1,2,3, \ldots, 10)$ give the results noted by the numbers written in the various order neighborhoods along the invariant directions $x_2 = 0$ and $x_3 = 0$. For example, the application of transformation V eight times along the invariant direction $x_2 = 0$ resulted in a simple point in the eighth order neighborhood of O_1. The sequence of eight ones along $x_2 = 0$ indicates that the multiplicity of the curves C_i was one in each of the neighborhoods from one to eight.

CHAPTER VII

A STUDY OF BRANCH POINT O_8'

Branch Point O_8'

To point $O_3(0,0,1)$ on the plane corresponds on F the point $O_8'(0,0,0,0,0,0,0,1)$. The singularities of O_8' will now be investigated. Let us consider the family (∞^6) of curves from the system $|C_0|$, passing thru the invariant point O_3. The equation of this family of curves is

$$(22)\quad a_1x_1^{11} + a_2x_1^8x_2^2x_3 + a_3x_1^5x_2^4x_3 + a_4x_1^4x_2^6x_3 + a_5x_1^2x_2^3x_3^6 + a_6x_1^7x_2^3x_3^5 + a_7x_2^{11} = 0.$$

Consider the transformation

$$M,\qquad x_1 : x_2 : x_3 = z_1z_2 : z_2z_3 : z_3^2$$

Applying the transformation M three times to equation (22), we get

$$(23)\quad z_3^{33}(a_5z_1^2+a_6z_1z_2+a_7z_2^2) + a_1z_1^{11}z_2^{24} + a_2z_1^8z_2^{16}z_3^{11} + a_3z_1^5z_2^8z_3^{22} + a_4z_1^4z_2^9z_3^{22} = 0.$$

This shows that the double point $(z_1 = z_2 = 0)$ corresponds to the point O_{3222} in the third order neighborhood of O_3 in the $x_1 = 0$ direction. We note that O_{32}, O_{322}, and O_{3222} are all double points.

In order to obtain the points, infinitely near $0_8'(0,0,0,0,0,0,0,1)$ on F, which correspond to the points infinitely near the point $(z_1 = z_2 = 0)$, it is necessary, first, to project the surface F from the point $0_8'$ onto the hyperplane $X_8 = 0$. This gives the independent equations

$$F_3, \quad X_1X_3 = X_2^2$$
$$X_2X_5 = X_3^2$$
$$X_3X_6 = X_4X_5$$
$$X_5X_7 = X_6^2 \quad \text{and } X_8 = 0,$$

and the dependent equations

$$X_1X_5 = X_2X_3$$
$$X_1X_6 = X_2X_4$$
$$X_2X_6 = X_3X_4$$
$$X_3X_7 = X_4X_6.$$

The order of the surface is 6 found by Coble's Method. Now, apply the transformation M three times to

$$\frac{X_1}{x_1^{11}} = \frac{X_2}{x_1^8x_2x_3^2} = \frac{X_3}{x_1^5x_2^2x_3^4} = \frac{X_4}{x_1^4x_2^6x_3} = \frac{X_5}{x_1^2x_2^3x_3^6} = \frac{X_6}{x_1x_2^7x_3^3} = \frac{X_7}{x_2^{11}}$$

and obtain the simplified expression

$$\frac{X_1}{z_1^{11}z_2^{24}} = \frac{X_2}{z_1^8z_2^{16}z_3^{11}} = \frac{X_3}{z_1^5z_2^8z_3^{22}} = \frac{X_4}{z_1^4z_2^9z_3^{22}} = \frac{X_5}{z_1^2z_3^{33}} = \frac{X_6}{z_1z_2z_3^{33}} = \frac{X_7}{z_2^2z_3^{33}}.$$

Since we are interested in approaching the point $(z_1 = z_2 = 0)$ from all directions, let $z_2 = kz_1$, and substitute in the last equations. This gives us

$$\frac{X_1}{k^{24}z_1^{33}} = \frac{X_2}{k^{16}z_1^{24}z_3^{11}} = \frac{X_3}{k^{8}z_1^{13}z_3^{22}} = \frac{X_4}{k^{9}z_1^{13}z_3^{22}} = \frac{X_5}{z_1^{2}z_3^{33}} = \frac{X_6}{kz_1^{2}z_3^{33}} = \frac{X_7}{k^{2}z_1^{2}z_3^{33}}$$

which simplifies to

$$\frac{X_1}{k^{24}z_1^{33}} = \frac{X_2}{k^{16}z_1^{22}z_3^{11}} = \frac{X_3}{k^{8}z_1^{11}z_3^{22}} = \frac{X_4}{k^{9}z_1^{11}z_3^{22}} = \frac{X_5}{z_3^{33}} = \frac{X_6}{kz_3^{33}} = \frac{X_7}{k^{2}z_3^{33}}.$$

Passing to the limit as z_1 goes to zero, the equations above will have meaning if $X_1 = X_2 = X_3 = X_4 = 0$. This gives

$$\frac{X_5}{z_3^{33}} = \frac{X_6}{kz_3^{33}} = \frac{X_7}{k^{2}z_3^{33}}.$$

Multiplying thru by z_3^{33}, we get

$$\frac{X_5}{1} = \frac{X_6}{k} = \frac{X_7}{k^2}, \quad X_1 = X_2 = X_3 = X_4 = 0. \tag{24}$$

It follows that

$$k = \frac{X_6}{X_5} = \frac{X_7}{X_6}, \quad X_1 = X_2 = X_3 = X_4 = 0.$$

Eliminating k, one gets

$$X_5X_7 = X_6^2$$
$$X_1 = X_2 = X_3 = X_4 = 0,$$

which is a quadric cone in S_7, tangent to F at O_8'.

We note that, since $X_8 = 0$,

$$X_5X_7 = X_6^2$$
$$X_1 = X_2 = X_3 = X_4 = X_8 = 0$$

is a conic in S_7.

Now the quadric cone intersects F_3 in

$$(G_3) \qquad X_5X_7 = X_6^2,$$
$$X_1 = X_2 = X_3 = X_4 = X_8 = 0$$

which is the above conic in S_7 on the tangent cone. This shows that the points in the first order neighborhood of the point $(z_1 = z_2 = 0)$ as well as the points in the first order neighborhood of 0_{3222}, correspond projectively to the points on this conic.

When we apply the quadratic transformation

$$N, \qquad X_1 : X_2 : X_3 = z_1z_3 : z_1z_2 : z_3^2$$

two times to the curve (22), we get

$$z_3^{22}(a_1z_1^3 + a_2z_1^2z_2 + a_3z_1z_2^2 + a_5z_2^3) + a_4z_1^{8\,6\,11}z_2^{2\,3} + a_6z_1^{7\,7\,11}z_2^{2\,3} + a_7z_1^{14\,11}z_2^{3} = 0.$$

This shows that to 0_{311} in the second order neighborhood of 0_3 on $x_2 = 0$ corresponds triply the point $(z_1 = z_2 = 0)$.

Apply the transformation twice to the curves of the x-plane and then project the resultant curves of the z-plane over onto S_7 by means of

$$\frac{X_1}{x_1^{11}} \cdot \frac{X_2}{x_1^{8}x_2^{2}} \cdot \frac{X_3}{x_1^{5\,2}x_2^{4}} \cdot \frac{X_4}{x_1^{4\,6}x_2^{3}} \cdot \frac{X_5}{x_1^{2\,3\,6}x_2^{3}} \cdot \frac{X_6}{x_1^{7\,3}x_2^{3}} \cdot \frac{X_7}{x_2^{11}}$$

and

$$\frac{X_1}{z_1^{3\,22}z_3^{22}} \cdot \frac{X_2}{z_1^{2}z_2^{22}z_3^{22}} \cdot \frac{X_3}{z_1^{2}z_2^{2}z_3^{22}} \cdot \frac{X_4}{z_1^{8\,6\,11}z_2^{2}z_3^{3}} \cdot \frac{X_5}{z_2^{3\,22}z_3^{2}} \cdot \frac{X_6}{z_1^{7\,7\,11}z_2^{2}z_3^{2}} \cdot \frac{X_7}{z_1^{14\,11}z_2^{2}}.$$

Again let $z_2 = kz_1$ and substitute. One gets

$$\frac{X_1}{z_1^3 z_3^{22}} - \frac{X_2}{kz_1^3 z_3^{22}} - \frac{X_3}{k^2 z_1^3 z_3^{22}} - \frac{X_4}{k^6 z_1^{14} z_3^{11}} - \frac{X_5}{k^3 z_1^3 z_3^{22}} - \frac{X_6}{k^7 z_1^{14} z_3^{11}} - \frac{X_7}{k^{11} z_1^{25}}$$

or

$$\frac{X_1}{z_3^{22}} - \frac{X_2}{kz_3^{22}} - \frac{X_3}{k^2 z_3^{22}} - \frac{X_4}{k^6 z_1^{11} z_3^{11}} - \frac{X_5}{k^3 z_3^{22}} - \frac{X_6}{k^7 z_1^{11} z_3^{11}} - \frac{X_7}{k^{11} z_1^{22}}.$$

When allowing z_1 to approach zero, this reduces to

(25)
$$\frac{X_2}{X_1} = \frac{X_3}{X_2} = \frac{X_5}{X_3} = k, \quad X_4 = X_6 = X_7 = 0,$$

or

$$X_1 X_3 = X_2^2,$$
$$X_2 X_5 = X_3^2,$$
$$X_4 = X_6 = X_7 = 0.$$

Thus in S_7 we have a cubic cone tangent to F at $0_8'$. It intersects F_3 in

(C_4)
$$X_1 X_3 = X_2^2,$$
$$X_2 X_5 = X_3^2,$$
$$X_4 = X_6 = X_7 = X_8 = 0,$$

which is a cubic space curve in S_7 on the cubic tangent cone. This shows that the points in the first order neighborhood of the point ($z_1 = z_2 = 0$) as well as the points in the first order neighborhood of 0_{311} correspond projectively to the points of this cubic curve.

It has now been established that to the invariant point 0_3 of the involution I_{11}, corresponds on the surface F a branch point $0_8'$

which is a 5-tuple point and at which is a quadric cone and a cubic cone tangent to F. The curves G_3 and G_4 have in common the point $X_1 = X_2 = X_3 = X_4 = X_6 = X_7 = X_8 = 0$, designated by $O_5'(0,0,0,0,1,0,0,0)$.

Images of Curves $|C_1|$ at O_8'

Consider the system of curves

$$|C_1|, \quad b_1x_1^{10}x_2 + b_2x_1^7x_2^2x_3^2 + b_3x_1^4x_2^3x_3^4 + b_4x_1^3x_2^7x_3 + b_5x_1^2x_3^9 + b_6x_1x_2^4x_3^6 + b_7x_2^8x_3^3 = 0.$$

Applying the transformation M three times gives

$$(26) \quad z_3^{33}(b_5z_1^2+b_6z_1z_2+b_7z_2^2) + b_1z_1^{10}z_2^{25} + b_2z_1^7z_2^{17}z_3^{11} + b_3z_1^4z_2^9z_3^{22} + b_4z_1^3z_2^{10}z_3^{22} = 0.$$

The curve (26) has a double point at the point $(z_1 = z_2 = 0)$, and the tangent to the curve at this point has the equation

$$b_5z_1 + b_6z_1z_2 + b_7z_2^2 = 0.$$

Thus the point O_{3222}, in the third order neighborhood of O_3, is a double point, and there are two simple variable points in the first order neighborhood of O_{3222}. We also note that O_{32} and O_{322} are double points.

To approach the point $(z_1 = z_2 = 0)$ along the curve (26), substitute $z_2 = kz_1$ and allow z_1 to approach zero. The results are

$$z_3^{33}(b_5z_1^2+b_6kz_1^2+b_7k^2z_1^2) + b_1k^{25}z_1^{35} + b_2k^{17}z_1^{24}z_3^{11} + b_3k^9z_1^{13}z_3^{22}$$

$$+ b_4k^{10}z_1^{13}z_3^{22} = 0$$

or

$$z_3^{33}(b_5+b_6k+b_7k^2) + b_1k^{25}z_1^{33} + b_2k^{17}z_1^{22}z_3^{11} + b_3k^9z_1^{11}z_3^{22}$$

$$+ b_4k^{10}z_1^{11}z_3^{22} = 0.$$

Hence,

$$b_5 + b_6k + b_7k^2 = 0.$$

The coordinates, then, of the two points on G_3, corresponding to the two variable points, infinitely near O_{3222}, are therefore

$$(0, 0, 0, 0, 1, k_a, k_a^2, 0) \text{ and}$$

$$(0, 0, 0, 0, 1, k_b, k_b^2, 0)$$

where k_a and k_b are the roots of $b_5 + b_6k + b_7k^2 = 0$.

Thus a member of the $|\Sigma_1|$ curves on F in S_7, which are the images of the curves $|C_1|$, has a double point at O_8', and the two tangent lines to Σ_1, determined by O_8' and the above points are represented by

$$X_1 = X_2 = X_3 = X_4 = 0$$

$$b_5X_5 + b_6X_6 + b_7X_7 = 0$$

$$-k_aX_6 + X_7 = 0$$

and

$$X_1 = X_2 = X_3 = X_4 = 0$$

$$b_5X_5 + b_6X_6 + b_7X_7 = 0$$

$$-k_bX_6 + X_7 = 0.$$

The point O_8' is a double point for the Σ_1 curves. They pass twice thru O_8' along elements of the quadric tangent cone.

Images of Curves $|C_2|$ at O_8'

Consider the system of curves

$$|C_2|, \quad c_1x_1^9x_2^2 + c_2x_1^6x_2^3x_3^2 + c_3x_1^4x_3^7 + c_4x_1^3x_2^4x_3^4 + c_5x_1^2x_2^8x_3 + c_6x_1x_2^9x_3^9 + c_7x_2^5x_3^6 = 0.$$

Applying the transformation M three times gives

$$(27) \quad z_3^{33}(c_6z_1+c_7z_2) + c_1z_1^9z_2^{25} + c_2z_1^6z_2^{17}z_3^{11} + c_3z_1^4z_2^8z_3^{22} + c_4z_1^3z_2^9z_3^{22} + c_5z_1^2z_2^{10}z_3^{22} = 0.$$

Thus the curve (27) has a simple point at the point $(z_1 = z_2 = 0)$ and the tangent to this curve at this point has for its equation $c_6z_1 + c_7z_2 = 0$. By a treatment similar to that for $|C_1|$ one obtains $c_6 + c_7k = 0$.

The points O_{32}, O_{322}, and O_{3222} are simple, and there is one simple variable point in the first order neighborhood of O_{3222}.

The value of k from this equation, substituted into equation (24), locates the point on G_3, corresponding to the point on $|C_2|$, in the first order neighborhood of O_{3222}. The coordinates of this point on G_3 are

$$(0, 0, 0, 0, c_7^2, -c_6c_7, c_6^2, 0).$$

A member of the curves $|\Sigma_2|$ on F in S_7 passes thru the point $0_8'(0,0,0,0,0,0,0,1)$ along the tangent line determined by the above point and $0_8'$. This tangent line is represented by

$$X_1 = X_2 = X_3 = X_4 = 0$$
$$c_6X_5 + c_7X_6 = 0$$
$$-c_6^2X_5 + c_7^2X_7 = 0.$$

Applying the transformation N twice gives

$$(28) \quad z_3^{22}(c_3z_1+c_6z_2) + c_1z_1^{10}z_2^{2}z_3^{11} + c_2z_1^{9}z_2^{3}z_3^{11} + c_4z_1^{8}z_2^{4}z_3^{11} + c_5z_1^{15}z_2^{8} + c_7z_1^{7}z_2^{5}z_3^{11} = 0.$$

Thus the curve (28) has a simple point at the point $(z_1 = z_2 = 0)$ and the tangent to this curve at this point has for its equation $c_3z_1 + c_6z_2 = 0$. By a treatment similar to that for $|C_1|$, one obtains $c_3 + c_6k = 0$.

The points 0_{31} and 0_{311} are simple, and there is a simple variable point in the first order neighborhood of 0_{311}.

The value of k from $c_3 + c_6k = 0$, substituted into equation (25), locates the point on G_4, corresponding to the point on $|C_2|$ in the first order neighborhood of 0_{311}. The coordinates of this point on G_4 are

$$(c_6^3, -c_3c_6^2, c_3^2c_6, 0, -c_3^3, 0, 0, 0).$$

A member of the curves $|\Sigma_2|$ on F in S_7 passes thru the point $0_8'$ along the tangent line determined by the above point and $0_8'$. This tangent line is represented by

$$X_4 = X_6 = X_7 = 0$$

$$c_5X_1 + c_6X_2 = 0$$

$$c_5X_3 + c_6X_5 = 0$$

$$c_5X_2 + c_6X_3 = 0.$$

Hence O_8' is a double point for the Σ_2 curves. One branch goes thru O_8' along an element of the cubic tangent cone, while the other branch goes thru along an element of the quadric cone.

Images of Curves $|C_3|$ at O_8'

Consider the system of curves

$$|C_3|, \quad d_1x_1^8x_2^3 + d_2x_1^6x_3^5 + d_3x_1^5x_2^4x_3^2 + d_4x_1^3x_2^1x_3^7 + d_5x_1^2x_2^5x_3^4 + d_6x_1x_2^9x_3 + d_7x_2^2x_3^9 = 0.$$

Applying the transformation N twice gives

$$(29) \quad z_3^{22}(d_2z_1^2+d_4z_1z_2+d_7z_2^2) + d_1z_1^{10}z_2^3z_3^{11} + d_3z_1^9z_2^4z_3^{11} + d_5z_1^8z_2^5z_3^{11} + d_6z_1^{15}z_2^9 = 0.$$

The curve (29) has a double point at the point $(z_1 = z_2 = 0)$, and the tangent to the curve at this point has the equation $d_2z_1^2 + d_4z_1z_2 + d_7z_2^2 = 0$. By a treatment similar to that for $|C_1|$ one obtains $d_2 + d_4k + d_7k^2 = 0$.

Thus the point O_{311} in the second order neighborhood of O_3, is a double point, and there are two simple variable points in the first order neighborhood of O_{311}.

The coordinates of the two points on C_4, corresponding to the two variable points infinitely near O_{311}, are therefore

$$(1,\ k_a,\ k_a^2,\ 0,\ k_a^3,\ 0,\ 0,\ 0) \quad \text{and}$$

$$(1,\ k_b,\ k_b^2,\ 0,\ k_b^3,\ 0,\ 0,\ 0)$$

where k_a and k_b are the roots of $d_2 + d_4k + d_7k^2 = 0$.

A member of the curves $|\Sigma_3|$, the images of the curves $|C_3|$ on F in S_7, has therefore a double point at O_8', and the two tangent lines to the curves at this point lie on the cubic cone

$$X_1X_3 = X_2^2, \quad X_2X_5 = X_3^2,$$

$$X_4 = X_6 = X_7 = 0,$$

and have for their equations

$$X_4 = X_6 = X_7 = 0$$

$$d_2X_1 + d_4X_2 + d_7X_3 = 0$$

$$k_aX_3 - X_5 = 0$$

$$k_aX_2 - X_3 = 0$$

and

$$X_4 = X_6 = X_7 = 0$$

$$d_2X_1 + d_4X_2 + d_7X_3 = 0$$

$$k_bX_3 - X_5 = 0$$

$$k_bX_2 - X_3 = 0$$

where the values of k are the roots of

$$d_2 + d_4k + d_7k^2 = 0.$$

Hence O_8' is a double point for the Σ_3 curve. It passes twice thru O_8' along elements of the cubic tangent cone.

Images of Curves $|C_4|$ at O_8'

Consider the system of curves

$$|C_4|, \quad e_1x_1^8x_3^3 + e_2x_1^7x_2^4 + e_3x_1^5x_2^5x_3^5 + e_4x_1^4x_2^5x_3^2 + e_5x_1^2x_2^2x_3^7 + e_6x_1^6x_2^4x_3 + e_7x_2^{10}x_3 = 0.$$

Applying the transformation M three times gives

$$(30) \quad z_3^{22}(e_5z_1^2+e_6z_1z_2+e_7z_2^2) + e_1z_1^8z_2^{16} + e_2z_1^7z_2^{17} + e_3z_1^5z_2^8z_3^{11} + e_4z_1^4z_2^9z_3^{11} = 0.$$

This curve has a double point at the point $(z_1 = z_2 = 0)$, and the tangent to this curve at this point has the equation $e_5z_1^2 + e_6z_1z_2 + e_7z_2^2 = 0$. By a treatment similar to that for $|C_1|$, one obtains $e_5 + e_6k + e_7k^2 = 0$.

Thus the point O_{3222} in the third order neighborhood of O_3, is a double point, and there are two simple variable points in the first order neighborhood of O_{3222}. We also note that O_{32} and O_{322} are double points.

The coordinates, then, of the two points on C_3, corresponding to the two variable points, infinitely near O_{3222}, are therefore

$$(0, 0, 0, 0, 1, k_a, k_a^2, 0) \quad \text{and}$$
$$(0, 0, 0, 0, 1, k_b, k_b^2, 0)$$

where k_a and k_b are the roots of

$$e_5 + e_6k + e_7k^2 = 0.$$

A member of the curves $|\Sigma_4|$, the images of the curves $|C_4|$ on F in S_7, has therefore a double point at O_8', and the two tangent lines to the curve at this point lie on the quadric cone

$$X_5X_7 = X_8^2, \qquad X_1 = X_2 = X_3 = X_4 = 0$$

and have for equations

$$X_1 = X_2 = X_3 = X_4 = 0$$
$$e_5X_5 + e_6X_6 + e_7X_7 = 0$$
$$-k_aX_6 + X_7 = 0$$

and

$$X_1 = X_2 = X_3 = X_4 = 0$$
$$e_5X_5 + e_6X_6 + e_7X_7 = 0$$
$$-k_bX_6 + X_7 = 0$$

where the values of k are the roots of $e_5 + e_6k + e_7k^2 = 0$.

Applying the transformation N twice gives

$$(31) \qquad z_3^{22}(e_1z_1^2+e_3z_1z_2+e_5z_2^2) + e_2z_1^9z_2^4z_3^{11} + e_4z_1^8z_2^5z_3^{11} + e_6z_1^7z_2^6z_3^{11} + e_7x_1^{14}x_2^{10} = 0.$$

A member of (31) has a double point at the point $(z_1 = z_2 = 0)$, and the tangent to the curve at this point has the equation $e_1z_1^2 + e_3z_1z_2 + e_5z_2^2 = 0$. By a treatment similar to that for $|C_1|$, one obtains $e_1 + e_3k + e_5k^2 = 0$.

Thus the point O_{311}, in the second order neighborhood of O_3, is a double point, and there are two simple variable points in the first order neighborhood of O_{311}. The point O_{31} is also a double point.

The coordinates of the two points on G_4, corresponding to the two variable points, infinitely near O_{511}, are therefore

$$(1,\ k_a,\ k_a^2,\ 0,\ k_a^3,\ 0,\ 0,\ 0) \text{ and}$$

$$(1,\ k_b,\ k_b^2,\ 0,\ k_b^3,\ 0,\ 0,\ 0)$$

where k_a and k_b are the roots of $e_1 + e_3k + e_5k^2 = 0$.

A member of the curves $|\Sigma_4|$, the images of the curves $|C_4|$ on F in S_7, has therefore a double point at O_8', and the two tangent lines to the curve at this point lie on the cubic cone

$$X_1X_3 = X_2^2, \quad X_2X_5 = X_3^2, \quad X_4 = X_6 = X_7 = 0$$

and have for equations

$$X_4 = X_6 = X_7 = 0$$

$$e_1X_1 + e_3X_2 + e_5X_3 = 0$$

$$-k_aX_3 + X_5 = 0$$

$$-k_aX_2 + X_3 = 0$$

and

$$X_4 = X_6 = X_7 = 0$$

$$e_1X_1 + e_3X_2 + e_5X_3 = 0$$

$$-k_bX_3 + X_5 = 0$$

$$-k_bX_2 + X_3 = 0$$

where the values of k are the roots of $e_1 + e_3k + e_5k^2 = 0$.

Hence O_8' is a 4-tuple point for the Σ_4 curve. One branch passes twice thru O_1' along two lines of the cubic tangent cone, while the second branch passes thru twice along two lines of the quadric tangent cone.

Images of Curves $|C_5|$ at O_8'

Consider the system of curves

$$|C_5|, \quad f_1x_1^{10}x_3 + f_2x_1^7x_2x_3^3 + f_3x_1^6x_2^5 + f_4x_1^4x_2^2x_3^5 + f_5x_1^3x_2^6x_3^2 + f_6x_1x_2^3x_3^7 + f_7x_2^7x_3^4 = 0.$$

Applying the transformation M three times gives

$$(32) \quad z_3^{33}(f_6z_1+f_7z_2) + f_1z_1^{10}z_2^{24} + f_2z_1^7z_2^{16}z_3^{11} + f_3z_1^6z_2^{17}z_3^{11} + f_4z_1^4z_2^8z_3^{22} + f_5z_1^3z_2^9z_3^{22} = 0.$$

Thus the curve (32) has a simple point at the point $(z_1 = z_2 = 0)$ and the tangent to this curve at this point has for its equation $f_6z_1 + f_7z_2 = 0$. By a treatment similar to that for $|C_1|$, one obtains $f_5 + f_7k = 0$.

The points O_{32}, O_{322}, and O_{3222} are simple, and there is one simple variable point in the first order neighborhood of O_{3222}.

The value of k from this equation substituted into equations (24), locates the point on G_3, corresponding to the point on $|C_5|$, in the first order neighborhood of O_{3222}. The coordinates of this point on G_3 are

$$(0, 0, 0, 0, f_7^2, -f_6f_7, f_6^2, 0).$$

A member of the curves $|\Sigma_5|$ on F in S_7 passes thru the point $O_8'(0,0,0,0,0,0,0,1)$ along the tangent line determined by the above point and O_8'. This tangent line is represented by

$$X_1 = X_2 = X_3 = X_4 = 0$$
$$f_6X_5 + f_7X_6 = 0$$
$$f_6X_6 + f_7X_7 = 0.$$

Applying the transformation N twice to the curve (32) gives

(33) $$z_3^{11}(f_1z_1^3+f_2z_1^2z_2+ f_4z_1z_2^2+f_6z_2^3) + f_3z_1^9z_2^5 + f_5z_1^8z_2^6 + f_7z_1^7z_2^7 = 0.$$

This curve has a triple point at the point $(z_1 = z_2 = 0)$, and the tangent to the curve at this point has the equation $f_1z_1^3 + f_2z_1^2z_2 + f_4z_1z_2^2 + f_6z_2^3 = 0$. By a treatment similar to that for $|C_1|$, one obtains $f_1 + f_2k + f_4k^2 + f_6k^3 = 0$.

Thus, the point O_{311}, in the second order neighborhood of O_3 is triple. Hence, there are three simple variable points in the first order neighborhood of O_{311}.

The coordinates of the three points on G_4, corresponding to the three variable points, infinitely near O_{311}, are therefore

$$(Q_1)\ (1,\ k_a,\ k_a^2,\ 0,\ k_a^3,\ 0,\ 0,\ 0)$$
$$(Q_2)\ (1,\ k_b,\ k_b^2,\ 0,\ k_b^3,\ 0,\ 0,\ 0)$$
$$(Q_3)\ (1,\ k_c,\ k_c^2,\ 0,\ k_c^3,\ 0,\ 0,\ 0),$$

where k_a, k_b, and k_c are the roots of above cubic equation.

These three points determine the plane

$$X_5 + aX_3 + bX_2 + cX_1 = 0, \quad X_4 = X_6 = X_7 = X_8 = 0$$

whose coefficients will be evaluated below. Substituting the roots in the equation for the plane gives

$$k_a^3 + ak_a^2 + bk_a + c = 0$$

$$k_b^3 + ak_b^2 + bk_b + c = 0$$

$$k_c^3 + ak_c^2 + bk_c + c = 0.$$

When these equations are solved simultaneously for a, b, and c, one gets

$$a = -(k_a + k_b + k_c)$$

$$b = k_ak_b + k_ak_c + k_bk_c$$

$$c = -k_ak_bk_c .$$

By making use of the relationship between roots and coefficients of rational integral equations, it is clear from

(34) $$f_1 + f_2k + f_4k^2 + f_6k^3 = 0 \quad \text{or}$$

$$k^3 + \frac{f_4}{f_6}k^2 + \frac{f_2}{f_6}k + \frac{f_1}{f_6} = 0,$$

that $$\frac{f_4}{f_6} = -(k_a + k_b + k_c) = a$$

and $$\frac{f_2}{f_6} = k_ak_b + k_ak_c + k_bk_c = b$$

and $$\frac{f_1}{f_6} = -k_ak_bk_c = c.$$

So the equation of the plane, $Q_1Q_2Q_3$, can now be written as

(35) $$X_4 = X_6 = X_7 = X_8 = 0, \qquad X_5 + \frac{f_4}{f_6}X_3 + \frac{f_2}{f_6}X_2 + \frac{f_1}{f_6}X_1 = 0$$

or $$f_1X_1 + f_2X_2 + f_4X_3 + f_6X_5 = 0, \quad X_4 = X_6 = X_7 = X_8 = 0.$$

These results show that a member of the image curves, designated by $|\Sigma_5|$, of the curves $|C_5|$ mapped upon the surface F, has three branches along elements of the cubic tangent cone. The tangents to the curve at O_8' are the intersections of the cubic cone (25) with the hyperplane

$$f_1X_1 + f_2X_2 + f_4X_3 + f_6X_5 = 0.$$

The equations of these tangents are

$$\begin{aligned} X_4 = X_6 = X_7 &= 0 \\ f_1X_1 + f_2X_2 + f_4X_3 + f_6X_5 &= 0 \\ -k_aX_2 + X_3 &= 0 \\ -k_aX_3 + X_5 &= 0 \end{aligned}$$

and

$$\begin{aligned} X_4 = X_6 = X_7 &= 0 \\ f_1X_1 + f_2X_2 + f_4X_3 + f_6X_5 &= 0 \\ -k_bX_2 + X_3 &= 0 \\ -k_aX_3 + X_5 &= 0 \end{aligned}$$

and

$$\begin{aligned} X_4 = X_6 = X_7 &= 0 \\ f_1X_1 + f_2X_2 + f_4X_3 + f_6X_5 &= 0 \\ -k_cX_2 + X_3 &= 0 \\ -k_cX_3 + X_5 &= 0. \end{aligned}$$

When a member of the curves $|\Sigma_5|$ is projected from O_8' upon the cubic surface F_3, the equations for the curve, now designated by $|\Sigma_5'|$, become

$$\begin{Vmatrix} X_1 & X_2 & X_3 & X_4 & X_4X_6 & b_2X_2 + b_4X_3 + b_6X_5 + b_7X_6 \\ X_2 & X_3 & X_5 & X_6 & X_5X_7 & - b_1X_2 - b_3X_4 - b_5X_6 \end{Vmatrix} = 0$$

$X_8 = 0.$

Hence O_8' is a 4-tuple point for the Σ_5 curves. One branch passes three times thru O_8' along elements of the cubic tangent cone, while the other branch passes thru once along an element of the quadric tangent cone.

Images of Curves $|C_6|$ at O_8'

Consider the system of curves

$$|C_6|, \quad g_1x_1^9x_2x_3 + g_2x_1^6x_2^2x_3^3 + g_3x_1^5x_2^6 + g_4x_1^3x_2^3x_3^5 + g_5x_1^2x_2^7x_3^2 + g_6x_1x_3^{10} + g_7x_2^4x_3^7 = 0.$$

Applying the transformation M three times gives

$$(36) \qquad z_3^{33}(g_6z_1+g_7z_2) + g_1z_1^9z_2^{25} + g_2z_1^6z_2^{17}z_3^{11} + g_3z_1^5z_2^{18}z_3^{11} + g_4z_1^3z_2^9z_3^{22} + g_5z_1^2z_2^{10}z_3^{22} = 0.$$

Thus the curve (36) has a simple point at the point $(z_1 = z_2 = 0)$ and the tangent to this curve at this point has for its equation $g_6z_1 + g_7z_2 = 0$. By a treatment similar to that for $|C_1|$, one obtains $g_6 + g_7k = 0$.

The points O_{32}, O_{322}, and O_{3222} are simple, and there is one simple variable point in the first order neighborhood of O_{3222}.

The value of k from this equation, substituted into equation (24), locates the point on G_3, corresponding to the point on $|C_6|$, in the first order neighborhood of O_{3222}. The coordinates of this point on G_3 are

$$(0,\ 0,\ 0,\ 0,\ g_7^2,\ -g_6g_7,\ g_6^2,\ 0).$$

The curve Σ_6 on F in S_7 passes thru the point $O_8'(0,0,0,0,0,0,0,1)$ along the tangent line determined by the above point O_8'. This tangent line is represented by

$$X_1 = X_2 = X_3 = X_4 = 0$$
$$g_6X_5 + g_7X_6 = 0$$
$$g_6X_6 + g_7X_7 = 0.$$

Hence O_8' is a simple point for the Σ_6 curve. It passes thru O_8' along an element of the quadric tangent cone.

Images of Curves $|C_7|$ at O_8'

Consider the system of curves

$$|C_7|,\quad h_1x_1^8x_2^2x_3 + h_2x_1^5x_2^3x_3^3 + h_3x_1^4x_2^7 + h_4x_1^3x_3^8 + h_5x_1^2x_2^4x_3^5 + h_6x_1x_2^8x_3^2 + h_7x_2x_3^{10} = 0.$$

Applying the transformation N twice gives

$$(37)\quad z_3^{22}(h_4z_1+h_7z_2) + h_1z_1^{10}z_2^2z_3^{11} + h_2z_1^9z_2^3z_3^{11} + h_3z_1^{16}z_2^7 + h_5z_1^8z_2^4z_3^{11} + h_6z_1^{15}z_2^8 = 0.$$

Thus the curve (37) has a simple point at the point $(z_1 = z_2 = 0)$ and the tangent to this curve at this point has for its equation $h_4 z_1 + h_7 z_2 = 0$. By a treatment similar to that for $|C_1|$, one obtains $h_4 + h_7 k = 0$.

The points O_{31} and O_{311} are simple, and there is one simple variable point in the first order neighborhood of O_{311}.

The value of k from $h_4 + kh_7 = 0$, substituted into equation (25), locates the point on G_4, corresponding to the point on $|C_7|$, in the first order neighborhood of O_{311}. The coordinates of this point on G_4 are

$$(h_7^3,\ -h_4 h_7^2,\ h_4^2 h_7,\ 0,\ -h_4^3,\ 0,\ 0,\ 0).$$

The curve Σ_7 on F in S_7 passes thru the point O_8' along the tangent line determined by the above point and O_8'. This tangent line is represented by

$$X_4 = X_6 = X_7 = 0$$
$$h_4 X_1 + h_7 X_2 = 0$$
$$h_4 X_3 + h_7 X_5 = 0$$
$$h_4 X_2 + h_7 X_3 = 0.$$

Hence O_8' is a simple point for the Σ_7 curve. It passes thru O_8' along an element of the cubic tangent cone.

Images of Curves $|C_8|$ at O_8'

Consider the system of curves

$$|C_8|, \quad j_1 x_1^7 x_2^3 x_3 + j_2 x_1^5 x_3^6 + j_3 x_1^4 x_2^4 x_3^3 + j_4 x_1^3 x_2^8 + j_5 x_1^2 x_2 x_3^8 + j_6 x_1 x_2^5 x_3^5 + j_7 x_2^9 x_3^2 = 0.$$

Applying the transformation M three times gives

$$(38)\qquad z_3^{22}(j_5z_1^2+j_6z_1z_2+j_7z_2^2) + j_1z_1^7z_2^{17} + j_2z_1^5z_2^8z_3^{11} + j_3z_1^4z_2^9z_3^{11}$$
$$+ j_4z_1^3z_2^{10}z_3^{11} = 0.$$

This curve has a double point at the point $(z_1 = z_2 = 0)$ and the tangent to the curve at this point has the equation $j_5z_1^2 + j_6z_1z_2 + j_7z_2^2 = 0$. By a treatment similar to that for $|C_1|$, one obtains $j_5 + j_6k + j_7k^2 = 0$.

Thus the point O_{3222}, in the third order neighborhood of O_3, is a double point, and there are two simple variable points in the first order neighborhood of O_{3222}. We also note that O_{32} and O_{322} are double points.

The coordinates, then, of the two points on G_5, corresponding to the two variable points, infinitely near O_{3222}, are therefore

$$(0, 0, 0, 0, 1, k_a, k_a^2, 0)$$
$$(0, 0, 0, 0, 1, k_b, k_b^2, 0)$$

where k_a and k_b are the roots of $j_5 + j_6k + j_7k^2 = 0$.

A member of the curves $|\Sigma_8|$, the images of the curves $|C_8|$ on F in S_7, has therefore a double point at O_8', and the two tangent lines to the curve at this point lie on the quadric cone

$$X_5X_7 = X_6^2, \qquad X_1 = X_2 = X_3 = X_4 = 0$$

and have for equations

$$X_1 = X_2 = X_3 = X_4 = 0$$
$$j_5X_5 + j_6X_6 + j_7X_7 = 0$$
$$-k_aX_6 + X_7 = 0$$

and

$$X_1 = X_2 = X_3 = X_4 = 0$$
$$j_5X_5 + j_6X_6 + j_7X_7 = 0$$
$$-k_bX_6 + X_7 = 0$$

where the values of k are the roots of $j_5 + j_6k + j_7k^2 = 0$.

Applying the transformation N twice gives

$$(39) \qquad z_3^{22}(j_2z_1 + j_5z_2) + j_1z_1^{9}z_2^{3}z_3^{11} + j_3z_1^{8}z_2^{4}z_3^{11} + j_4z_1^{15}z_2^{8} + j_8z_1^{7}z_2^{5}z_3^{11}$$
$$+ j_7z_1^{14}z_2^{9} = 0.$$

Thus the curve (39) has a simple point at the point $(z_1 = z_2 = 0)$, and the tangent to this curve at this point has for its equation $j_2z_1 + j_5z_2 = 0$. By a treatment similar to that for $|C_1|$, one obtains $j_2 + j_5k = 0$.

The points O_{31} and O_{311} are simple, and there is one simple variable point in the first order neighborhood of O_{311}.

The value of k, substituted into equation (25), locates the point on G_4, corresponding to the point on $|C_8|$, in the first order neighborhood of O_{311}. The coordinates of the point on G_4 are

$$(j_5^3, -j_2j_5^2, j_2^2j_5, 0, -j_2^3, 0, 0, 0).$$

The curve Σ_8 on F in S_7 passes thru the point O_8' along the tangent line determined by the above point on G_4 and O_8'. This tangent line is represented by

$$X_4 = X_6 = X_7 = 0$$
$$j_2X_1 + j_5X_2 = 0$$
$$j_2X_3 + j_5X_7 = 0$$
$$j_2X_2 + j_5X_3 = 0.$$

Hence O_8' is a triple point for the Σ_8 curve. One branch passes thru O_8' twice along elements of the quadric tangent cone, while the other branch passes thru along an element of the cubic tangent cone.

Images of Curves $|C_9|$ at O_8'

Consider the system of curves

$$|C_9|, \quad k_1x_1^7x_3^4 + k_2x_1^6x_2^4x_3 + k_3x_1^4x_2x_3^6 + k_4x_1^3x_2^5x_3^3 + k_5x_1^2x_2^9 + k_6x_1x_2^2x_3^8 + k_7x_2^6x_3^5 = 0.$$

Applying the transformation M three times gives

$$(40) \quad z_3^{22}(k_6z_1+k_7z_2) + k_1z_1^7z_2^{16} + k_2z_1^6z_2^{17} + k_3z_1^4z_2^8z_3^{11} + k_4z_1^3z_2^9z_3^{11} + k_5z_1^2z_2^{10}z_3^{11} = 0.$$

Thus the curve (40) has a simple point at the point $(z_1 = z_2 = 0)$ and the tangent to this curve at this point has for its equation $k_6z_1 + k_7z = 0$. By a treatment similar to that for $|C_1|$, one obtains $k_6 + k_7k = 0$.

The points O_{32}, O_{322}, and O_{3222} are simple, and there is one simple variable point in the first order neighborhood of O_{3222}.

The values of k from this equation, substituted into equation (24), locates the point on G_3, corresponding to the point on $|C_9|$, in the first order neighborhood of O_{3222}. The coordinates of this point on G_3 are

$$(0,\ 0,\ 0,\ 0,\ k_7^2,\ -k_6k_7,\ k_6^2,\ 0).$$

The curve Σ_9 on F in S_7 passes thru the point O_8' along the tangent line determined by the above point and O_8'. This tangent line is represented by

$$X_1 = X_2 = X_3 = X_4 = 0$$
$$k_6X_5 + k_7X_6 = 0$$
$$k_6X_6 + k_7X_7 = 0.$$

Applying the transformation N twice to the curve (40) gives

$$(41)\qquad z_3^{22}(k_1z_1^2+k_3z_1z_2+k_6z_2^2) + k_2z_1^9z_2^4z_3^{11} + k_4z_1^8z_2^5z_3^{11} + k_5z_1^{15}z_2^9 + k_7z_1^7z_2^8z_3^{11} = 0.$$

This curve has a double point at the point $(z_1 = z_2 = 0)$, and the tangent to the curve at this point has the equation $k_1z_1^2 + k_3z_1z_2 + k_6z_2^2 = 0$. By a treatment similar to that for $|C_1|$, one obtains $k_1 + k_3k + k_6k^2 = 0$.

The point O_{311}, in the second order neighborhood of O_3, is a double point, and there are two simple variable points in the

first order neighborhood O_{311}. We note that O_{31} is also a double point.

The coordinates, then, of the two points on O_4, corresponding to the two variable points, infinitely near O_{311}, are therefore

$$(1,\ k_a,\ k_a^2,\ 0,\ k_a^3,\ 0,\ 0,\ 0)$$

$$(1,\ k_b,\ k_b^2,\ 0,\ k_b^3,\ 0,\ 0,\ 0)$$

where k_a and k_b are the roots of $k_1 + k_3k + k_6k^2 = 0$.

The curve Σ_9, has therefore a triple point at O_8'. Two of the branches have two tangent lines which lie on the cubic cone

$$X_1X_3 = X_2^2$$
$$X_2X_5 = X_3^2$$
$$X_4 = X_6 = X_7 = 0.$$

Their equations are

$$X_4 = X_6 = X_7 = 0$$
$$k_1X_1 + k_3X_2 + k_6X_3 = 0$$
$$-k_aX_3 + X_5 = 0$$
$$-k_aX_2 + X_3 = 0$$

and

$$X_4 = X_6 = X_7 = 0$$
$$k_1X_1 + k_3X_2 + k_6X_3 = 0$$
$$-k_bX_3 + X_5 = 0$$
$$-k_bX_2 + X_3 = 0$$

where the values of k are the roots of $k_1 + k_3k + k_6k^2 = 0$.

Hence O_8' is a triple point for the Σ_9 curve. One branch passes thru O_8' along an element of the quadric tangent cone, while a second branch passes twice thru O_8' along elements of a cubic tangent cone.

Images of Curves $|C_{10}|$ at O_8'

Consider the system of curves

$$|C_{10}|, \quad l_1x_1^9x_3^2 + l_2x_1^6x_2x_3^4 + l_3x_1^5x_2^5x_3 + l_4x_1^3x_2^2x_3^6 + l_5x_1^2x_2^6x_3^3 + l_6x_1x_2^{10} + l_7x_2^3x_3^8 = 0.$$

Applying the transformation N twice gives

$$z_3^{22}(l_1z_1^3+l_2z_1^2z_2+l_4z_1z_2^2+l_7z_2^3) + l_3z_1^9z_2^5z_3^{11} + l_5z_1^8z_2^6z_3^{11} + l_6z_1^{15}z_2^{10} = 0. \tag{42}$$

This curve has a triple point at the point $(z_1 = z_2 = 0)$, and the tangent to the curve at this point has the equation

$$l_1z_1^3 + l_2z_1^2z_2 + l_4z_1z_2^2 + l_7z_2^3 = 0.$$

By a similar treatment to that for $|C_1|$, one gets

$$l_1 + l_2k + l_4k^2 + l_7k^3 = 0. \tag{43}$$

Thus the point O_{311}, in the second order neighborhood of O_3 is triple. Hence there are three simple variable points in the first order neighborhood of O_{311}.

The points in the third order neighborhood of O_3 project into the curve G_4. Any one member of the system $|C_{10}|$ has three points in the third order neighborhood of O_3, and their projections on F_3 are therefore on G_4. The three values of k, found by solving equation (43), will therefore give the values of k in the parametric equations (25), which will locate the three points on G_4.

Assume that the roots of (43) are k_a, k_b, and k_c.

The coordinates of the points on G_4 are

$$(Q_1) \quad (1, k_a, k_a^2, 0, k_a^3, 0, 0, 0)$$
$$(Q_2) \quad (1, k_b, k_b^2, 0, k_b^3, 0, 0, 0)$$
$$(Q_3) \quad (1, k_c, k_c^2, 0, k_c^3, 0, 0, 0).$$

The points Q_1, Q_2, and Q_3 determine a plane, as explained in the study of the curve (32), whose equation is

$$l_1X_1 + l_2X_2 + l_4X_3 + l_7X_5 = 0, \quad X_4 = X_6 = X_7 = X_8 = 0.$$

These results show that a member of the images, designated by $|z_{10}|$, of the curves $|C_{10}|$ mapped upon the surface F, has a triple point at O_8', and the tangents to the curve at this point are the intersections of the cubic cone (25) and the hyperplane

$$l_1X_1 + l_2X_2 + l_4X_3 + l_7X_5 = 0.$$

The equations of these tangents are

$$X_4 = X_6 = X_7 = 0$$
$$l_1X_1 + l_2X_2 + l_4X_3 + l_7X_5 = 0$$
$$-k_aX_3 + X_5 = 0$$
$$-k_aX_2 + X_3 = 0,$$

$$X_4 = X_6 = X_7 = 0$$
$$l_1X_1 + l_2X_2 + l_4X_3 + l_7X_5 = 0$$
$$-k_bX_3 + X_5 = 0$$
$$-k_bX_2 + X_3 = 0,$$

and

$$X_4 = X_6 = X_7 = 0$$
$$l_1X_1 + l_2X_2 + l_4X_3 + l_7X_5 = 0$$
$$-k_cX_3 + X_5 = 0$$
$$-k_bX_2 + X_3 = 0.$$

Hence O_8' is a triple point for the Σ_{10} curves. They pass three times thru O_8' along elements of a cubic tangent cone.

Summary

In this chapter, as in the previous chapter, ten invariant systems of plane algebraic curves of degree eleven were mapped onto the surface F. Their images, $|\Sigma_i|$ (i = 1,2,3, . . . , 10), were studied at the branch point $O_8'(0,0,0,0,0,0,0,1)$, where O_8' was the image of O_3. The tangent elements to the surface F in S_7 at O_8' were a quadric cone and a cubic cone.

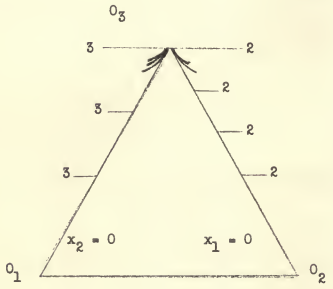
$|\Sigma_1|$, $|\Sigma_2|$, and $|\Sigma_3|$ passed doubly thru O_8'. $|\Sigma_1|$ had two tangent lines on the quadric cone; $|\Sigma_2|$ had one tangent line on the quadric cone and the other on the cubic cone, while $|\Sigma_3|$ had two tangent lines on the cubic cone.

$|\Sigma_4|$ and $|\Sigma_5|$ passed four times thru O_8'. $|\Sigma_4|$ had two tangent lines on the quadric cone and two on the cubic cone, while $|\Sigma_5|$ had one tangent line on the quadric cone and three on the cubic cone.

$|\Sigma_6|$ and $|\Sigma_7|$ passed simply thru O_8'. $|\Sigma_6|$ had one tangent line on the quadric cone, while $|\Sigma_7|$ had one on the cubic cone.

$|\Sigma_8|$, $|\Sigma_9|$, and $|\Sigma_{10}|$ passed three times thru $0_8'$. $|\Sigma_8|$ had two tangent lines on the quadric cone and one on the cubic cone. $|\Sigma_9|$ had one tangent line on the quadric cone and two on the cubic cone, while $|\Sigma_{10}|$ had three tangent lines on the cubic cone.

The reference triangle for this study is



CHAPTER VIII

A STUDY OF BRANCH POINT O_7'

Branch Point O_7'

To the point $O_2(0,1,0)$ on the plane corresponds on F the point $O_7'(0,0,0,0,0,0,1,0)$. The singularities of this point will now be studied. Let us consider the family (∞^6) of curves from the system $|C_0|$, passing thru the invariant point O_2. The equation of this family of curves is

$$(44)\quad a_1x_1^{11} + a_2x_1^{8}x_2x_3^{2} + a_3x_1^{5}x_2^{2}x_3^{4} + a_4x_1^{4}x_2^{6}x_3 + a_5x_1^{2}x_2^{3}x_3^{6} + a_6x_1^{7}x_2^{3} + a_8x_3^{11} = 0.$$

Consider the transformation

$$L,\qquad x_1 : x_2 : x_3 = z_1z_2 : z_2^2 : z_1z_3$$

Applying the transformation L six times we get

$$(45)\quad z_2^{66}(a_1z_1+a_4z_3) + a_2z_1^{10}z_2^{55}z_3^{2} + a_3z_1^{19}z_2^{44}z_3^{4} + a_5z_1^{28}z_2^{33}z_3^{6} + a_6z_1^{9}z_2^{55}z_3^{5} + a_8z_1^{56}z_3^{11} = 0.$$

This shows that the simple point $(z_1 = z_3 = 0)$, corresponds to the point $O_{2111111}$ in the sixth order neighborhood of O_2 in the $x_3 = 0$ direction. We note that O_{21} is double while O_{211}, O_{2111}, $\cdots$, O_{211111} are all simple points.

In order to obtain the points, infinitely near O_7' on F, which correspond to the points infinitely near the point $(z_1 = z_3 = 0)$, it is necessary, first, to project the surface F from the point O_7' onto the hyperplane $X_7 = 0$. This gives the independent equations

F_2, $$X_4X_8 = X_5^2$$
$$X_2X_5 = X_3^2$$
$$X_1X_3 = X_2^2$$
$$X_3X_6 = X_4X_5 \text{ and } X_7 = 0,$$

and the dependent equations

$$X_1X_5 = X_2X_3$$
$$X_1X_6 = X_2X_4$$
$$X_2X_6 = X_3X_4$$
$$X_3X_6X_8 = X_5^3.$$

The order of F_2 is 8 as found by Coble's Method.

Now apply the L transformation six times to

$$\frac{X_1}{x_1^{11}} \cdot \frac{X_2}{x_1^8 x_2^2} \cdot \frac{X_3}{x_1^5 x_2^2 x_3^4} \cdot \frac{X_4}{x_1^4 x_2^6} \cdot \frac{X_5}{x_1^2 x_2^3 x_3^6} \cdot \frac{X_6}{x_1 x_2^7 x_3^3} \cdot \frac{X_8}{x_3^{11}}$$

and obtain the simplified expression

$$\frac{X_1}{z_1^{66} z_2^{66}} \cdot \frac{X_2}{z_1^{10} z_2^{55} z_3^2} \cdot \frac{X_3}{z_1^{19} z_2^{44} z_3^4} \cdot \frac{X_4}{z_2^{66} z_3^{66}} \cdot \frac{X_5}{z_1^{28} z_2^{33} z_3^6} \cdot \frac{X_6}{z_1^9 z_2^{55} z_3^3} \cdot \frac{X_8}{z_1^{56} z_3^{11}}.$$

Since we are interested in approaching the point $(z_1 = z_3 = 0)$ from all directions, let $z_3 = kz_1$ and substitute in the last equations. This gives

$$\frac{X_1}{z_1 z_2^{66}} - \frac{X_2}{k^2 z_1^{12} z_2^{55}} - \frac{X_3}{k^4 z_1^{23} z_2^{44}} - \frac{X_4}{k z_1 z_2^{66}} - \frac{X_5}{k^6 z_1^{34} z_2^{33}} - \frac{X_6}{k^3 z_1^{12} z_2^{55}} - \frac{X_8}{k^{11} z_1^{67}}$$

which simplifies to

$$\frac{X_1}{z_2^{66}} - \frac{X_2}{k^2 z_1^{11} z_2^{55}} - \frac{X_3}{k^4 z_1^{22} z_2^{44}} - \frac{X_4}{k z_2^{66}} - \frac{X_5}{k^6 z_1^{66}} - \frac{X_6}{k^3 z_1^{11} z_2^{55}} - \frac{X_8}{k^{11} z_1^{66}}.$$

Passing to the limit as z_1 approaches zero, the equations above will have meaning if $X_2 = X_3 = X_5 = X_6 = X_8 = 0$. This gives $\frac{X_1}{1} = \frac{X_4}{k}$. Thus we get

(46) $X_4 = kX_1, \quad X_2 = X_3 = X_5 = X_6 = X_8 = 0,$

which is a plane in S_7 tangent to surface F at point $0_7'$.

The equations

$(b_1), \quad X_4 = kX_1, \quad X_2 = X_3 = X_5 = X_6 = X_7 = X_8 = 0$

represent a straight line in S_7 on the tangent plane. This shows that the points in the first order neighborhood of the point $(z_1 = z_3 = 0)$, as well as the points in the first order neighborhood of $0_{2111111}$ correspond projectively to the points of this line. This line does not pass thru $0_7'$ in the tangent plane.

Let us now apply the transformation L once, followed by the transformation K† to (44). We get

(46A) $$z_2^{22}(a_4 z_1 + a_6 z_3) + a_1 z_1^7 z_2^{11} z_3^5 + a_2 z_1^6 z_2^{11} z_3^6 + a_3 z_1^5 z_2^{11} z_3^7 + a_5 z_1^4 z_2^{11} z_3^8 + a_8 z_1^7 z_3^{16} = 0.$$

†K is given on page 99.

This shows that the simple point $(z_1 = z_3 = 0)$, corresponds to the point 0_{213} in the second order neighborhood of 0_2. We note that 0_{21} is a double point.

We now apply the transformation L once, followed by the transformation K to

$$\frac{X_1}{x_1^{11}} - \frac{X_2}{x_1^8 x_2 x_3^2} - \frac{X_3}{x_1^5 x_2^2 x_3^4} - \frac{X_4}{x_1^4 x_2^6 x_3} - \frac{X_5}{x_1^2 x_2^5 x_3^6} - \frac{X_6}{x_1 x_2^4 x_3^5} - \frac{X_8}{x_3^{11}}$$

and obtain the simplified expression

$$\frac{X_1}{z_1^7 z_2^{11} z_3^5} - \frac{X_2}{z_1^6 z_2^{11} z_3^6} - \frac{X_3}{z_1^5 z_2^{11} z_3^7} - \frac{X_4}{z_1 z_2^{22}} - \frac{X_5}{z_1^4 z_2^{11} z_3^8} - \frac{X_6}{z_2^{22} z_3} - \frac{X_8}{z_1^7 z_3^{16}}.$$

Since we are interested in approaching the point $(z_1 = z_3 = 0)$ from all directions, let $z_3 = kz_1$, and substitute in the last equations. This gives

$$\frac{X_1}{k^5 z_1^{12} z_2^{11}} - \frac{X_2}{k^6 z_1^{12} z_2^{11}} - \frac{X_3}{k^7 z_1^{12} z_2^{11}} - \frac{X_4}{z_1 z_2^{22}} - \frac{X_5}{k^8 z_1^{12} z_2^{11}} - \frac{X_6}{k z_1 z_2^{22}} - \frac{X_8}{k^{16} z_1^{23}},$$

which simplifies to

$$\frac{X_1}{k^5 z_1^{11} z_2^{11}} - \frac{X_2}{k^6 z_1^{11} z_2^{11}} - \frac{X_3}{k^7 z_1^{11} z_2^{11}} - \frac{X_4}{z_2^{22}} - \frac{X_5}{k^8 z_1^{11} z_2^{11}} - \frac{X_6}{k z_2^{22}} - \frac{X_8}{k^{16} z_1^{22}}.$$

Passing to the limit as z_1 approaches zero, the equations above will have meaning if

(46B) $\quad X_1 = X_2 = X_3 = X_5 = X_8 = 0 \quad$ and $X_6 = kX_4$.

These equations define a second plane in S_7 which is tangent to the surface F at $0_7'$.

The equations

$$(b_2), \qquad X_6 = kX_4, \qquad X_1 = X_2 = X_3 = X_5 = X_7 = X_8 = 0$$

represent a straight line in S_7 on this tangent plane. This shows that the points in the first order neighborhood of the point $(z_1 = z_3 = 0)$ as well as the points in the first order neighborhood of 0_{213}, correspond projectively to the points of this line.

Thus we see that these two tangent planes at $0_7'$ are cut by the hyperplane $X_7 = 0$ in two lines, b_1 and b_2. These two lines intersect in the point $0_4'(0,0,0,1,0,0,0,0)$. Hence the line of intersection of the two tangent planes is the line determined by the two points $0_7'$ and $0_4'$. Any point on this line has the coordinates $(0,0,0,1,0,0,\varepsilon,0)$. The equation of this line is

$$X_1 = X_2 = X_3 = X_5 = X_6 = X_8 = 0.$$

When we apply the quadratic transformation

$$K, \qquad x_1 : x_2 : x_3 = z_1 z_3 : z_2^2 : z_2 z_3$$

seven times to the curves (44), we get, after simplifying,

$$z_2^{77}(a_6 z_1 + a_8 z_3) + a_1 z_1^{11} z_3^{67} + a_2 z_1^{8} z_2^{22} z_3^{48} + a_3 z_1^{5} z_2^{44} z_3^{29} + a_4 z_1^{4} z_2^{55} z_3^{19}$$

$$+ a_5 z_1^{2} z_2^{66} z_3^{10} = 0.$$

This shows that to $0_{23333333}$ in the seventh order neighborhood of 0_2 on $x_1 = 0$ corresponds simply the point $(z_1 = z_3 = 0)$. In referring to a point such as $0_{23333333}$, we shall designate it as $0_{23(7)}$.

Applying the quadratic transformation K to the curves of the x-plane and then projecting the resultant curves of the z-plane over onto S_7 by means of

$$\frac{X_1}{x_1^{11}} \cdot \frac{X_2}{x_1^{8}x_2^{2}} \cdot \frac{X_3}{x_1^{5}x_2^{2}x_3^{4}} \cdot \frac{X_4}{x_1^{4}x_2^{6}} \cdot \frac{X_5}{x_1^{2}x_2^{3}x_3^{6}} \cdot \frac{X_6}{x_1x_2x_3^{7}} \cdot \frac{X_8}{x_3^{11}}$$

and get

$$\frac{X_1}{z_1^{11}z_3^{67}} \cdot \frac{X_2}{z_1^{8}z_2^{22}z_3^{48}} \cdot \frac{X_3}{z_1^{5}z_2^{44}z_3^{29}} \cdot \frac{X_4}{z_1^{4}z_2^{55}z_3^{19}} \cdot \frac{X_5}{z_1^{2}z_2^{66}z_3^{10}} \cdot \frac{X_6}{z_1^{77}z_2^{77}} \cdot \frac{X_8}{z_2^{77}z_3^{77}}.$$

Let $z_3 = kz_1$ and substitute. One gets

$$\frac{X_1}{k^{67}z_1^{78}} \cdot \frac{X_2}{k^{48}z_1^{56}z_2^{22}} \cdot \frac{X_3}{k^{29}z_1^{34}z_2^{44}} \cdot \frac{X_4}{k^{19}z_1^{23}z_2^{55}} \cdot \frac{X_5}{k^{10}z_1^{12}z_2^{66}} \cdot \frac{X_6}{z_1^{77}z_2^{77}} \cdot \frac{X_8}{kz_1^{77}z_2^{77}}$$

or

$$\frac{X_1}{k^{67}z_1^{77}} \cdot \frac{X_2}{k^{48}z_1^{55}z_2^{22}} \cdot \frac{X_3}{k^{29}z_1^{33}z_2^{44}} \cdot \frac{X_4}{k^{19}z_1^{22}z_2^{55}} \cdot \frac{X_5}{k^{10}z_1^{11}z_2^{66}} \cdot \frac{X_6}{z_2^{77}} \cdot \frac{X_8}{kz_2^{77}}.$$

When we allow z_1 to approach zero, this reduces to

(46c) $\quad kX_6 = X_8, \quad X_1 = X_2 = X_3 = X_4 = X_5 = 0.$

It follows, therefore, that the points of the straight line

$(b_3), \qquad X_1 = X_2 = X_3 = X_4 = X_5 = X_7 = 0, \quad kX_6 = X_8$

correspond to the points infinitely near the point $O_{23(7)}$.

This also means that the points infinitely near $O_{23(7)}$ correspond to the points infinitely near the point O_7' on F and lying in the plane $X_1 = X_2 = X_3 = X_4 = X_5 = 0, \quad kX_6 = X_8$.

This plane is tangent to F at the point O_7'.

It has now been established that to the invariant point O_2 of the involution I_{11}, corresponds on the surface F a branch point O_7', which is a 3-tuple point.

Images of Curves $|C_1|$ at O_7'

Consider now the study of the system of curves

$$|C_1|, \quad b_1x_1^{10}x_2 + b_2x_1^7x_2^2x_3 + b_3x_1^4x_2^3x_3^4 + b_4x_1^3x_2^7x_3 + b_5x_1^2x_3^9 + b_6x_1^4x_2^4x_3^6 + b_7x_2^8x_3^3 = 0.$$

A member of the system of curves represented by C_1 passes thru the invariant point O_2, which is triple for these curves. It is tangent to the invariant direction $x_3 = 0$.

Applying the transformation L six times to the system $|C_1|$ gives

$$(47) \quad z_2^{55}(b_1z_1+b_4z_3) + b_2z_1^{10}z_2^{44}z_3^2 + b_3z_1^{19}z_2^{33}z_3^4 + b_5z_1^{47}z_2^9 + b_6z_1^{28}z_2^{22}z_3^6 + b_7z_1^9z_2^{44}z_3^3 = 0.$$

This shows that to $O_{21(6)}$ in the sixth order neighborhood of O_2 on $x_3 = 0$ corresponds simply the point $(z_1 = z_3 = 0)$. We note that O_{211}, O_{2111}, . . . , $O_{21(5)}$ are all simple points, while O_{21} is double.

The tangent to this curve at this point has for its equation $b_1z_1 + b_4z_3 = 0$.

To approach the point along the curve (47), substitute $z_3 = kz_1$ into its equation and allow z_1 to approach zero. The results are

$$z_2^{55}(b_1z_1+b_4kz_1) + b_2k^2z_1^{12}z_2^{44} + b_3k^4z_1^{23}z_2^{33} + b_5z_1^{47}z_2^{9}$$

$$+ b_6k^6z_1^{34}z_2^{22} + b_7k^3z_1^{12}z_2^{44} = 0.$$

$$z_2^{55}(b_1+b_4k) + b_2k^2z_1^{11}z_2^{44} + b_3k^4z_1^{22}z_2^{33} + b_5z_1^{46}z_2^{9} + b_6k^6z_1^{33}z_2^{22}$$

$$+ b_7k^3z_1^{11}z_2^{44} = 0,$$

and

$$b_1 + b_4k = 0.$$

The value of k from this equation, substituted into equations (46), locates the point on the line b_1 corresponding to the point on $|C_1|$, in the first order neighborhood of $0_{21(6)}$. The coordinates of this point on b_1 are

$$(b_4, 0, 0, -b_1, 0, 0, 0, 0).$$

A member of the $|\Sigma_1|$ curves on F in S_7 passes thru the point $0_7'(0,0,0,0,0,0,1,0)$ along the tangent line determined by the above point and $0_7'$. This tangent line is represented by

$$b_1X_1 + b_4X_4 = 0$$

$$X_2 = X_3 = X_5 = X_6 = X_8 = 0.$$

Similarly, the coordinates of this point on b_2 are

$$(0, 0, 0, -b_7, 0, b_4, 0, 0),$$

and the resulting tangent line is

$$b_4X_4 + b_7X_6 = 0$$

$$X_1 = X_2 = X_3 = X_5 = X_8 = 0.$$

Hence we see that there are two tangent lines to Σ_1, passing thru $0_7'$, and lying one on each of two tangent planes.

We shall refer hereafter to these tangent planes, which are projectively equivalent to a neighborhood along $x_3 = 0$, as A and B. Also, the other tangent plane at O_7' will be called C.

Images of Curves $|C_2|$ at O_7'

Consider now the system of curves

$$|C_2|, \quad c_1x_1^9x_2^2 + c_2x_1^6x_2^3x_3^2 + c_3x_1^4x_3^7 + c_4x_1^3x_2^4x_3^4 + c_5x_1^2x_2^8x_3 + c_6x_1x_2^9x_3^2 + c_7x_2^5x_3^6 = 0.$$

A member of this system of curves represented by C_2 passes thru the invariant point O_2, which is triple for this curve, with tangent $x_1 = 0$ counted twice and tangent $x_3 = 0$.

Applying the transformation K two times and the transformation L once to the curve C_2 gives

$$(48) \quad z_2^{33}(c_5z_1+c_7z_3) + c_1z_1^{21}z_3^{13} + c_2z_1^{14}z_2^{11}z_3^9 + c_3z_1^{13}z_2^{11}z_3^{10} + c_4z_1^7z_2^{22}z_3^5 + c_6z_1^6z_2^{22}z_3^6 = 0.$$

This shows that to O_{2331} in the third order neighborhood of O_2 corresponds simply the point $(z_1 = z_3 = 0)$, and the tangent to this curve at this point has for its equation $c_5z_1 + c_7z_3 = 0$. By a treatment similar to that used for $|C_1|$, we obtain $c_5 + c_7k = 0$.

The value of k from this equation, substituted into equations (46C), locates the point on the line b_3 corresponding to the point on C_2, in the first order neighborhood of O_{2331}. The coordinates of this point on b_3 are

$$(0, 0, 0, 0, 0, c_7, 0, -c_5).$$

A member of the $|\Sigma_2|$ curves on F in S_7 passes thru the point $0_7'(0,0,0,0,0,0,1,0)$ along the tangent line determined by the above point and $0_7'$. This tangent line is represented by

$$c_5X_6 + c_7X_8 = 0$$

$$X_1 = X_2 = X_3 = X_4 = X_5 = 0.$$

Applying the transformation L six times to the curves $|C_2|$ gives

$$(49) \qquad z_2^{55}(c_1z_1+c_5z_3) + c_2z_1^{10}z_2^{44}z_3^{2} + c_3z_1^{38}z_2^{11}z_3^{7} + c_4z_1^{19}z_2^{33}z_3^{4}$$

$$+ c_6z_1^{47}z_3^{9} + c_7z_1^{28}z_2^{22}z_3^{6} = 0.$$

This shows that to $0_{21(6)}$ in the sixth order neighborhood of 0_2 corresponds simply the point $(z_1 = z_3 = 0)$, and the tangent to this curve at this point has for its equation $c_1z_1 + c_5z_3 = 0$. A treatment similar to that used for $|C_1|$ results in $c_1 + c_5k = 0$.

The value of k from this equation, substituted into equations (46), locates the point on the line b_1, corresponding to the point on C_2, in the first order neighborhood of $0_{21(6)}$. The coordinates of this point on b_1 are

$$(c_5, 0, 0, -c_1, 0, 0, 0, 0).$$

A member of the $|\Sigma_2|$ curves on F in S_7 passes thru the point $0_7'(0,0,0,0,0,0,1,0)$ along the tangent line determined by the above point and $0_7'$. The tangent line is represented by

$$c_1X_1 + c_5X_4 = 0$$

$$X_2 = X_3 = X_5 = X_6 = X_8 = 0.$$

Hence O_7' is a double point for the $|\Sigma_2|$ curves. One branch of each curve passes once thru O_7' along a line in the tangent plane C, while the second branch passes thru along a line in the tangent plane A.

Images of Curves $|C_3|$ at O_7'

Consider now the system of curves

$$|C_3|, \quad d_1x_1^8x_2^3 + d_2x_1^6x_3^5 + d_3x_1^5x_2^4x_3^2 + d_4x_1^3x_2^7x_3 + d_5x_1^2x_2^5x_3^4 + d_6x_1x_2^9x_3$$
$$+ d_7x_2^2x_3^9 = 0.$$

A member of this system of curves represented by C_3 passes thru the invariant point O_2, which is double for this curve, with tangents $x_1 = 0$ and $x_3 = 0$.

Applying the transformation K seven times to the curves $|C_3|$ gives

$$(50) \quad z_2^{55}(d_6z_1+d_7z_3) + d_1z_1^8z_3^{48} + d_2z_1^6z_2^{11}z_3^{39} + d_3z_1^5z_2^{22}z_3^{29} + d_4z_1^3z_2^{33}z_3^{20}$$
$$+ d_5z_1^2z_2^{44}z_3^{10} = 0.$$

This shows that to $O_{23(7)}$ in the seventh order neighborhood of O_2 corresponds simply the point $(z_1 = z_3 = 0)$, and the tangent to this curve at this point has for its equation $d_6z_1 + d_7z_3 = 0$. By a treatment similar to that used for $|C_1|$ we obtain $d_6 + d_7k = 0$.

The value of k from this equation, substituted into equations (46C) locates the point on the line b_3, corresponding to the point on $|C_3|$ in the first order neighborhood of $O_{23(7)}$. The coordinates of this point on b_3 are

$$(0, 0, 0, 0, 0, d_7, 0, -d_6).$$

A member of the $|\Sigma_3|$ curves on F in S_7 passes thru the point $O_7'(0,0,0,0,0,0,1,0)$ along the tangent line determined by the above point and O_7'. The tangent line is represented by

$$d_6X_6 + d_7X_8 = 0$$
$$X_1 = X_2 = X_3 = X_4 = X_5 = 0.$$

Applying the transformation L six times to the curve C_3 gives

$$(51) \quad z_2^{55}(d_1z_1+d_6z_3) + d_2z_1^{29}z_2^{22}z_3^5 + d_3z_1^{10}z_2^{44}z_3^2 + d_4z_1^{38}z_2^{11}z_3^7$$
$$+ d_5z_1^{19}z_2^{33}z_3^4 + d_7z_1^{47}z_3^9 = 0.$$

This shows that to $O_{21(6)}$ in the sixth order neighborhood of O_2 corresponds simply the point $(z_1 = z_3 = 0)$, and the tangent to this curve at this point has for its equation $d_1z_1 + d_6z_3 = 0$. By a treatment similar to that for $|C_1|$, we obtain $d_1 + d_6k = 0$.

The value of k from this equation, substituted into equations (46), locates the point on the line b_1, corresponding to the point on C_3 in the first order neighborhood of $O_{21(6)}$. The coordinates of this point on b_1 are

$$(d_6, 0, 0, -d_1, 0, 0, 0, 0).$$

A member of the $|\Sigma_3|$ curves on F in S_7 passes thru the point $O_7'(0,0,0,0,0,0,1,0)$ along the tangent lines determined by the above point and O_7'. The tangent lines are represented by

$$d_1X_1 + d_6X_4 = 0$$
$$X_2 = X_3 = X_5 = X_6 = X_8 = 0.$$

Hence O_7' is a double point for the $|\Sigma_3|$ curves. One branch of each curve passes thru O_7' along a line in tangent plane C, while the second branch passes thru along a line in the tangent plane A.

Images of Curves $|C_4|$ at O_7'

Consider now the system of curves

$$|C_4|, \quad e_1x_1^8x_3^3 + e_2x_1^7x_2^4 + e_3x_1^5x_2^5x_3^5 + e_4x_1^4x_2^5x_3^2 + e_5x_1^2x_2^2x_3^7 + e_6x_1^6x_2^4x_3 + e_7x_2^{10}x_3 = 0.$$

A member of the system $|C_4|$ passes thru the invariant point O_2, which is simple for this curve, with the tangent $x_3 = 0$.

Applying the transformation L six times to the curves $|C_4|$ gives

$$(52) \quad z_2^{44}(e_2z_1+e_7z_3) + e_1z_1^{20}z_2^{22}z_3^3 + e_3z_1^{29}z_2^{11}z_3^5 + e_4z_1^{10}z_2^{33}z_3^2 + e_5z_1^{38}z_3^7 + e_6z_1^{19}z_2^{22}z_3^4 = 0.$$

This shows that to $O_{21(6)}$ in the sixth order neighborhood of O_2 corresponds simply the point $(z_2 = z_3 = 0)$, and the tangent to these curves at this point has for its equation $e_2z_1 + e_7z_3 = 0$. By a treatment similar to that for $|C_1|$, one obtains $e_2 + e_7k = 0$.

The value of k from this equation, substituted into equations (46), locates the point on the line b_1, corresponding to the point on C_4 in the first order neighborhood of $O_{21(6)}$. The coordinates of this point on b_1 are

$$(e_7, 0, 0, -e_2, 0, 0, 0, 0).$$

A member of the $|\Sigma_4|$ curves on F in S_7 passes thru the point $O_7'(0,0,0,0,0,0,1,0)$ along the tangent line determined by the above point and O_7'. The tangent line is represented by

$$e_2X_1 + e_7X_4 = 0$$

$$X_2 = X_3 = X_5 = X_6 = X_8 = 0.$$

Hence O_7' is a simple point for a member of the $|\Sigma_4|$ curves. It passes thru O_7' along a line in the tangent plane A.

Images of Curves $|C_5|$ at O_7'

Consider now the system of curves

$$|C|, \quad f_1x_1^{10}x_3 + f_2x_1^7x_2x_3^3 + f_3x_1^6x_2^5 + f_4x_1^4x_2^2x_3^5 + f_5x_1^3x_2^6x_3^2 + f_6x_1x_2^3x_3^7 + f_7x_2^7x_3^4 = 0.$$

A member of the system $|C_5|$ passes thru the invariant point O_2, which is a 4-tuple point for this curve, with the tangent $x_3 = 0$ counted four times.

Applying first the transformation L and then the transformation K to the curves $|C_5|$ gives

$$(53) \qquad z_2^{11}(f_3z_1^2 + f_5z_1z_3 + f_7z_3^2) + f_1z_1^7z_3^6 + f_2z_1^6z_3^7 + f_4z_1^5z_3^8 + f_6z_1^4z_3^9 = 0.$$

This curve has a double point at the point $(z_1 = z_3 = 0)$, and the tangent to the curve at this point has the equation

$$f_3z_1^2 + f_5z_1z_3 + f_7z_3^2 = 0.$$

By a treatment similar to that for $|C_1|$, we get

$$f_7k^2 + f_5k + f_3 = 0.$$

Thus the point 0_{213}, in the second order neighborhood of 0_2, is a double point. Furthermore, we note that 0_{21} is a double point.

The coordinates, then, of the two points of b_2, corresponding to the two variable points, infinitely near 0_{213}, are therefore

$$(0, 0, 0, 1, 0, k_a, 0, 0)$$

and

$$(0, 0, 0, 1, 0, k_b, 0, 0)$$

where k_a and k_b are the roots of $f_7k^2 + f_5k + f_3 = 0$.

A member of the curves $|\Sigma_5|$, the images of curves $|C_5|$ on F in S_7, has therefore a double point at $0_7'$, and the tangent lines to the curve at this point have for their equations

$$X_6 = k_aX_4$$
$$X_1 = X_2 = X_3 = X_5 = X_8 = 0$$

and

$$X_6 = k_bX_4$$
$$X_1 = X_2 = X_3 = X_5 = X_8 = 0.$$

Hence $0_7'$ is a double point for a member of the $|\Sigma_5|$ curves. It passes twice thru $0_7'$, along two lines in the tangent plane B.

Images of Curves $|C_6|$ at O_7'

Consider the system of curves

$$|C_6|, \quad g_1x_1^9x_2x_3 + g_2x_1^6x_2^2x_3 + g_3x_1^5x_2^6 + g_4x_1^3x_2^3x_3^5 + g_5x_1^2x_2^7x_3^2 + g_6x_1x_3^{10} + g_7x_2^4x_3^7 = 0.$$

Applying the transformation K twice, followed by the transformation L, gives

$$(54) \quad z_2^{33}(g_5z_1+g_7z_3) + g_1z_1^{21}z_3^{13} + g_2z_1^{14}z_2^{11}z_3^9 + g_3z_1^{14}z_2^{10}z_3^{10} + g_4z_1^7z_2^{22}z_3^5 + g_6z_1^6z_2^{22}z_3^6 = 0.$$

Thus the curve (54) has a simple point at the point $(z_1 = z_3 = 0)$ and the tangent to this curve at this point has for its equation $g_5z_1 + g_7z_3 = 0$. By a treatment similar to that for $|C_1|$, we obtain $g_5 + g_7k = 0$.

The point O_{23} is double, while the points O_{233} and O_{2331} are simple, and there is one simple variable point in the first order neighborhood of O_{2331}.

The value of k from this equation, substituted into equations (46A), locates the point on b_3, corresponding to the point on $|C_6|$, in the first order neighborhood of O_{2331}. The coordinates of this point on b_3 are

$$(0, 0, 0, 0, 0, g_7, 0, -g_5).$$

The curves $|\Sigma_6|$ on F in S_7 pass thru the point $O_7'(0,0,0,0,0,0,1,0)$ along the tangent line determined by the above

point and O_7'. This tangent line is represented by

$$g_5X_6 + g_7X_8 = 0$$
$$X_1 = X_2 = X_3 = X_4 = X_5 = 0.$$

Applying the transformation L, followed by the transformation K, to $|C_6|$, gives

$$(55) \quad z_2^{22}(g_3z_1+g_5z_3) + g_1z_1^6z_2^{11}z_3^6 + g_2z_1^5z_2^{11}z_3^7 + g_4z_1^4z_2^{11}z_3^8 + g_6z_1^7z_3^{16} + g_7z_1^3z_2^{11}z_3^9 = 0.$$

Thus the curve (55) has a simple point at the point $(z_1 = z_3 = 0)$, and the tangent to this curve at this point has for its equation $g_3z_1 + g_5z_3 = 0$. By a treatment similar to that for $|C_1|$, we obtain $g_3 + g_5k = 0$.

The points O_{21} and O_{213} are simple, and there is one simple variable point in the first order neighborhood of O_{213}.

The value of k from this equation, substituted into equations (46B), locates the point on b_2 corresponding to the point on $|C_6|$, in the first order neighborhood of O_{213}. The coordinates of this point on b_2 are

$$(0, 0, 0, g_5, 0, -g_3, 0, 0).$$

A member of the curves $|\Sigma_6|$ on F in S_7 passes thru the point $O_7'(0,0,0,0,0,0,1,0)$ along the tangent line determined by the above point and O_7'. This tangent line is represented by

$$g_3X_4 + g_5X_6 = 0$$
$$X_1 = X_2 = X_3 = X_5 = X_8 = 0.$$

Hence O_7' is a double point for the Σ_6 curves. One branch passes thru O_7' along a line in tangent plane C, the second branch passes thru along a line in the tangent plane B.

Images of Curves $|C_7|$ at O_7'

Consider the system of curves

$$|C_7|, \quad h_1x_1^8x_2^2x_3 + h_2x_1^5x_2^3x_3^3 + h_3x_1^4x_2^7 + h_4x_1^3x_3^8 + h_5x_1^2x_2^4x_3^5 + h_6x_1^8x_2^2x_3 + h_7x_2x_3^{10} = 0.$$

Applying the transformation K seven times gives

$$(56) \quad z_2^{55}(h_6z_1+h_7z_3) + h_1z_1^8z_3^{48} + h_2z_1^5z_2^{22}z_3^{29} + h_3z_1^4z_2^{33}z_3^{19} + h_4z_1^3z_2^{33}z_3^{20} + h_5z_1^2z_2^{44}z_3^{10} = 0.$$

Thus the curve (56) has a simple point at the point $(z_1 = z_3 = 0)$, and the tangent to this curve at this point has for its equation $h_6z_1 + h_7z_3 = 0$. By a treatment similar to that for $|C_1|$, we obtain $h_6 + h_7k = 0$.

The points O_{23}, O_{233}, . . . , $O_{23(7)}$ are simple, and there is one simple variable point in the first order neighborhood of $O_{23(7)}$.

The value of k from this equation, substituted into equations (46C), locates the point on b_3, corresponding to the point on $|C_7|$, in the first order neighborhood of $O_{23(7)}$. The coordinates of this point on b_3 are

$$(0, 0, 0, 0, 0, g_5, 0, -g_3).$$

A member of the curves $|\Sigma_7|$ on F in S_7 passes thru the point $0_7'(0,0,0,0,0,0,1,0)$ along the tangent line determined by the above point and $0_7'$. This tangent line is represented by

$$g_3X_6 + g_5X_8 = 0$$
$$X_1 = X_2 = X_3 = X_4 = X_5 = 0.$$

Applying the transformation L and K, in succession, gives

$$(57) \quad z_2^{22}(h_3z_1+h_6z_3) + h_1z_1^6z_2^{11}z_3^6 + h_2z_1^5z_2^{11}z_3^7 + h_4z_1^8z_3^{15} + h_5z_1^4z_2^{11}z_3^8 + h_7z_1^7z_3^{16} = 0.$$

Thus the curve (57) has a simple point at the point $(z_1 = z_3 = 0)$, and the tangent to this curve at this point has for its equation $h_3z_1 + h_6z_3 = 0$. By a treatment similar to that for $|C_1|$, we get $h_3 + h_6k = 0$.

The points 0_{21} and 0_{213} are simple, and there is one simple variable point in the first order neighborhood of 0_{213}.

The value of k from this equation, substituted into equations for b_2, locates the point on b_2, corresponding to the point in $|C_7|$, in the first order neighborhood of 0_{213}. The coordinates of this point on b_2 are

$$(0, 0, 0, -h_6, 0, h_3, 0, 0)$$

and the equation of the resulting tangent line is

$$h_6X_6 + h_3X_4 = 0$$
$$X_1 = X_2 = X_3 = X_5 = X_8 = 0.$$

Hence O_7' is a double point for the Σ_7 curves. One branch passes thru O_7' along a line in plane C, while the second branch passes along a line in tangent plane B.

Images of Curves $|C_8|$ at O_7'

Consider the system of curves

$$|C_8|, \quad j_1x_1^7x_2^3x_3 + j_2x_1^5x_3^6 + j_3x_1^4x_2^4x_3^3 + j_4x_1^3x_2^8 + j_5x_1^2x_2x_3^8 + j_6x_1^5x_2^5x_3^5 + j_7x_2^9x_3^2 = 0.$$

Applying the transformations L and K in succession gives

$$(58) \quad z_2^{22}(j_4z_1+j_7z_3) + j_1z_1^6z_2^{11}z_3^6 + j_2z_1^9z_3^{14} + j_3z_1^5z_2^{11}z_3^7 + j_5z_1^8z_3^{15} + j_6z_1^4z_2^{11}z_3^8 = 0.$$

Thus the curve (58) has a simple point at the point $(z_1 = z_3 = 0)$ and the tangent to this curve at this point has for its equation $j_4z_1 + j_7z_3 = 0$. By a treatment similar to that for $|C_1|$, we obtain $j_4 + j_7k = 0$.

The points O_{21} and O_{213} are simple, and there is one simple variable point in the first order neighborhood of O_{213}.

The value of k from this equation, substituted into the equations for b_2, locates the point on b_2, corresponding to the point on C_8, in the first order neighborhood of O_{213}. The coordinates of this point on b_2 are

$$(0, 0, 0, -j_7, 0, j_4, 0, 0)$$

and the equations of the resulting tangent line are

$$j_7 x_6 + j_4 x_4 = 0$$

$$X_1 = X_2 = X_3 = X_5 = X_8 = 0.$$

Hence O_7' is a simple point for a member of the $|\Sigma_8|$ curves. It passes once thru O_7' along a line in the tangent plane B.

Images of Curves $|C_9|$ at O_7'

Consider the system of curves

$$|C_9|, \quad k_1 x_1^7 x_3^4 + k_2 x_1^6 x_2^4 x_3 + k_3 x_1^4 x_2 x_3^6 + k_4 x_1^5 x_2^5 x_3^3 + k_5 x_1^2 x_2^9 + k_6 x_1 x_2^2 x_3^8 + k_7 x_2^6 x_3^5 = 0.$$

Applying the transformations K twice and L once, gives

$$(59) \quad z_2^{33}(k_5 z_1 + k_7 z_3) + k_1 z_1^{20} z_3^{14} + k_2 z_1^{14} z_2^{11} z_3^9 + k_3 z_1^{15} z_2^{11} z_3^{10} + k_4 z_1^7 z_2^{22} z_3^5 + k_6 z_1^6 z_2^{22} z_3^6 = 0.$$

A member of the system (59) has a simple point at the point $(z_1 = z_3 = 0)$ and the tangent to this curve at this point has for its equation $k_5 z_1 + k_7 z_3 = 0$. By a treatment similar to that for $|C_1|$, we obtain $k_5 + k_7 k = 0$.

The points O_{23} and O_{233} are double, while O_{2331} is simple, and there is one simple variable point in the first order neighborhood of O_{2331}.

The value of k from this equation, substituted into equations (46C), locates the point on b_5, corresponding to the point on $|C_9|$, in the first order neighborhood of O_{2331}. The coordinates of this point on b_5 are

$$(0,\ 0,\ 0,\ 0,\ 0,\ k_7,\ 0,\ -k_5).$$

A member of the curves $|\Sigma_9|$ on F in S_7 passes thru the point $O_7'(0,0,0,0,0,0,1,0)$ along the tangent line determined by the above point and O_7'. These tangent lines are represented by

$$k_5X_6 + k_7X_8 = 0$$
$$X_1 = X_2 = X_3 = X_4 = X_5 = 0.$$

Hence O_7' is a simple point for a member of the $|\Sigma_9|$ curves. It passes once thru O_7' along a line in tangent plane C.

Images of Curves $|C_{10}|$ at O_7'

Consider the system of curves

$$|C_{10}|,\quad l_1x_1^9x_3^2 + l_2x_1^6x_2x_3^4 + l_3x_1^5x_2^5x_3 + l_4x_1^3x_2^2x_3^6 + l_5x_1^2x_2^6x_3^3 + l_6x_1x_2^{10} + l_7x_2^3x_3^8 = 0.$$

Applying the transformation K seven times gives

$$(60)\qquad z_2^{66}(l_6z_1+l_7z_3) + l_1x_1^9x_3^{58} + l_2z_1^6z_2^{22}z_3^{39} + l_3z_1^5z_2^{33}z_3^{29} + l_4z_1^3z_2^{44}z_3^{20} + l_5z_1^2z_2^{55}z_3^{10} = 0.$$

A member of the system (60) has a simple point at the point ($z_1 = z_3 = 0$) and the tangent to this curve at this point has for its equation $l_6 z_1 + l_7 z_3 = 0$. By a treatment similar to that for $|C_1|$, one obtains $l_6 + l_7 k = 0$.

The points O_{23}, O_{233}, . . . , $O_{23(7)}$ are all simple, and there is one simple variable point in the first order neighborhood of $O_{23(7)}$.

The value of k from this equation, substituted into equations (46A), locates the point on b_3, corresponding to the point on C_{10}, in the first order neighborhood of $O_{23(7)}$. The coordinates of this point on b_3 are

$$(0, 0, 0, 0, 0, l_7, 0, -l_6).$$

A member of the system $|\Sigma_{10}|$ on F in S_7 passes thru the point $O_7'(0,0,0,0,0,0,1,0)$ along the tangent line determined by the above point and O_7'. This tangent line is represented by

$$l_6 X_6 + l_7 X_8 = 0$$
$$X_1 = X_2 = X_3 = X_4 = X_5 = 0.$$

Hence O_7' is a simple point for a member of the $|\Sigma_{10}|$ curves. It passes thru O_7' along a line in tangent plane C.

Summary

In this chapter, as in the two previous chapters, we studied the system of image curves $|\Sigma_i|$ $(i = 1,2,3,\ldots,10)$.

The branch point O_7' on F, the image of O_2, was studied. O_7' was a 3-tuple point, while the tangent elements to the surface F in S_7 at O_7' were three planes, planes A and B corresponding to the $x_3 = 0$ direction, and plane C the $x_1 = 0$ direction.

$|\Sigma_1|$ passed doubly thru O_7' with one tangent line in each of the planes A and B.

$|\Sigma_2|$ and $|\Sigma_3|$ passed doubly thru O_7' with one tangent line in each of the planes A and C.

$|\Sigma_4|$ passed simply thru O_7' with the tangent line in plane A.

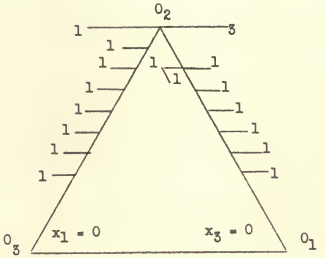
$|\Sigma_5|$ passed doubly thru O_7' with two tangent lines in plane B.

$|\Sigma_6|$ and $|\Sigma_7|$ passed doubly thru O_7', with one tangent in each of the planes B and C respectively.

$|\Sigma_8|$, $|\Sigma_9|$ and $|\Sigma_{10}|$ passed simply thru O_7', where $|\Sigma_8|$ had a tangent line in plane B, and $|\Sigma_9|$ and $|\Sigma_{10}|$ each had a tangent in plane C.

The author would not be surprised to find that this last chapter may some day be extended, possibly finding O_7' to have a fourth tangent plane. At present all efforts have only brought to light the three planes A, B and C.

The reference triangle for this study is



APPENDIX I

A METHOD FOR FINDING THE ORDER OF SURFACE F

We shall apply here the method of Coble for finding the order of a surface. We shall consider the surface F in S_7 for I_{11}.

$$F, \quad \left\| \begin{matrix} X_1 & X_2 & X_3 & X_4 & X_5^2 & X_4X_6 \\ X_2 & X_3 & X_5 & X_6 & X_6X_8 & X_5X_7 \end{matrix} \right\| = 0$$

Since F is a surface, it can have a parametric representation in terms of two parameters, which we shall designate as X_5 and t, where $X_1 = tX_2$.

$$X_1 = t^3X_5 \qquad X_5 = X_5$$

$$X_2 = t^2X_5 \qquad X_6 = 1$$

$$X_3 = tX_5 \qquad X_7 = \frac{1}{X_5}$$

$$X_4 = t \qquad X_8 = \frac{X_5^2}{t}$$

According to Coble's method, we now cut the surface with two hyperplanes.

1. $X_7 = A_1X_1 + B_1X_2 + C_1X_3 + D_1X_4 + E_1X_6$

2. $X_8 = A_2X_1 + B_2X_2 + C_2X_3 + D_2X_4 + E_2X_6$

When we substitute the parametric values for the X's in equations one and two, we get

3. $\frac{1}{X_5} = A_1t^3X_5 + B_1t^2X_5 + C_1tX_5 + D_1t + E_1$

4. $\frac{X_5^2}{t} = A_2t^3X_5 + B_2t^2X_5 + C_2tX_5 + D_2t + E_2$

After simplification, we get the following quadratic equations in X_5:

5. $(A_1t^3+B_1t^2+C_1t)X_5^2 + (D_1t+E_1)X_5 - 1 = 0$

6. $X_5^2 - (A_2t^4+B_2t^3+C_2t^2)X_5 + (-D_2t - E_2t) = 0.$

By use of the eliminant of equations five and six [3], we can derive an equation of degree eleven in t; hence the surface F is of order eleven.

APPENDIX II

A NOTE ON POINTS OF COINCIDENCE AND PERFECT POINTS

Let P(a,b,c,...,k) be any point on a surface F. Let C be any curve on surface F which passes thru P, and let $p_i = 0$, where $i = 1,\ldots,k-2$, be the equations for the tangent line to curve C at P. Furthermore, let T be a projective transformation under which F is invariant. If point P, under T, goes into itself, it is called a point of coincidence (under T).

Assume now that curve C is non-invariant under T. If, for every curve C thru point P, the tangent line $p_i = 0$ at P is transformed into itself under T, then point P is called a perfect point. Otherwise it is imperfect.

Finally, if a coincidence point is also perfect, the point is called a perfect coincidence point.

BIBLIOGRAPHY

1. Childress, Noel Arthur. "Surfaces Obtained from Involutions Generated by Homographies of Periods Three, Five, and Thirteen." Unpublished form of Ph.D. dissertation, Department of Mathematics, University of Florida, 1954.

2. Dessart, J. Memoires de la Societe Royale des Sciences de Liege, 3e Serie, Tome XVI (1931), pp. 1-28.

3. Dickson, Eugene. Theory of Equations. New York: John Wiley and Sons, Inc., 1939.

4. Godeaux, Lucien. Cours de Geometrie Superieure. Fascicule II. Liege: Librairie Bourguignon, 1937.

5. Godeaux, M. L. "Etude Elementaire sur l'Homographie Plane de Periode Trois et sur une Surface Cubique," Nouvelles Annales de Mathematiques, 4e Serie (1916), 16:49-61.

6. Godeaux, Lucien. Geometrie Algebrique. 2 tomes. Liege: Sciences et Lettres, 1948-1949.

7. Gormsen, Svend T. "Maps of Certain Algebraic Curves Invariant under Cyclic Involutions of Periods Three, Five, and Seven." Unpublished form of Ph.D. dissertation, Department of Mathematics, University of Florida, 1953.

8. Graustein, William C. Introduction to Higher Geometry. New York: The Macmillan Company, 1946.

9. Hodge and Pedoe. Methods of Algebraic Geometry. Volume I. Cambridge: The University Press, 1947.

10. Hutcherson, W. R. "A Cyclic Involution of Period Eleven," Canadian Journal of Mathematics, Vol. III (1951), pp. 155-158.

11. Hutcherson, W. R. et Childress, N.A. "Etude d'une involution cyclique de periode cinq," Bulletin Acad. Royale de Beligique, 6e Serie, Tome XL (1954), pp. 103-108.

12. Love, Clyde E. Differential and Integral Calculus. Fourth Edition. New York: The Macmillan Co., 1943.

13. Morelock, J. C. "Invariants With Respect to Special Projective Transformations." Unpublished form of Ph. D. dissertation, Department of Mathematics, University of Florida, 1952, pp. 26-30.

14. National Research Council, Bulletin Number 63. Selected Topics in Algebraic Geometry. N.R.C. of the National Academy of Sciences. Washington, D.C., 1928.

15. Olmstead, John M. H. Solid Analytic Geometry. New York: Appleton-Century-Crofts, Inc., 1947.

16. Snyder, Virgil, and Sisam, C. H. Analytic Geometry of Space. New York: Henry Holt and Company, 1914.

17. Walker, Robert J. Algebraic Curves. Princeton University Press, 1950.

18. Welchman, W. Gordon. Introduction to Algebraic Geometry. Cambridge: The University Press, 1950.

19. Woods, Frederick S. Higher Geometry. Boston: Ginn and Company, 1922.

BIOGRAPHICAL SKETCH

Stanley Frank was born on April 21, 1926 in New York City. He attended thirteen public schools in almost as many states. His first employment was that of magazine salesman at the age of seven. He has held thirty-four different positions from that time up to the present writing.

In 1944, the author entered City College on a tuition-free basis. During the next few years, while earning his living and furthering his academic growth, he attended the New York Phoenix School of Design and the New York Business School.

From 1947 to 1953, the author volunteered his services at the Lighthouse, the Guild for the Blind, two settlement houses, and four other welfare organizations. His main contribution during this period was the teaching of various skills to individuals and small groups.

In 1950, the author received the Bachelor of Science degree from City College, with majors in Mathematics and Education. From 1950 until 1953, the author taught grade school during the day and attended City College in the late afternoon, again on a scholarship basis, with the result that he obtained the degree of Master of Arts in 1953.

From 1953 to 1955, the author did graduate study in the College of Education at the University of Florida. In 1955, he accepted a graduate assistantship in the Department of Mathematics. He also

began a program of study, leading to the degree of Doctor of Philosophy, with Mathematics as the major field and Education as the minor.

The author has been a member of the National Education Association and the Mathematical Association of America.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

January 30, 1960

A H Gotaas
Dean, College of Arts and Sciences

Dean, Graduate School

SUPERVISORY COMMITTEE:

W. R. Hutchenson
Chairman

Andrew Sobczyk

T. O. Moore

D. E. Smith

Hal T. Lewis